H. Gold

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(Rossignol)
OKE

ELEMENTS

OF

GEOMETRY,

TRANSLATED FROM THE FRENCH OF

J. J. ROSSIGNOL,

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY
OF MILAN.

Quærentem dictis quibus
Clara tuæ possim præpandere lumina menti,
Res quibus occultas penitus convisere possis.

LUCR.

The SECOND EDITION.

LONDON:

PRINTED FOR J. JOHNSON, No. 72, ST. PAUL'S CHURCH-YARD.

M.DCC.LXXXVII.

ADVERTISEMENT.

THE Translator of this Work offers it to his Countrymen in their native tongue, because it appears to him to be one of the most successful attempts which has ever been made to facilitate the Study of Geometry. Originality of plan, clearness of arrangement, simplicity of demonstration, and an elegant conciseness of expression, will, he apprehends, be found to be the leading characters of these Elements.

Those who know where the difficulties of an undertaking of this kind principally lie, will easily pardon the liberties which the Author has taken, in making use of Numbers in the doctrine of Proportion, and, in employing the methods of Exhaustions and Indivisibles in demonstrating the properties of Solids, when they perceive that by this means he has enabled the learner to make himself master of the principal elementary propositions, with the most

perfect

perfect facility, and at the same time with a degree of accuracy sufficient for every purpose of practice. Nor will it be thought too bold an innovation, that in a mathematical work the Author makes use of Physical Points, *and these too of different magnitudes; if it be observed, that he always expresses by this term the idea commonly affixed to* Aliquot Parts, *and that he only applies it to commensurable lines or figures actually existing.*

It is not the Editor's idea, that this work can supersede the use of the Elements of Euclid, which have a strictness of proof, and universality of application, of which none of the more easy methods of demonstration, invented by moderns can boast. But he is of opinion, that it may enable those who are employed in the practical use of the Mathematical Arts, at an easy expence of time and attention, to acquire a general knowledge of the theory on which they are founded; and that it may be of great benefit to young Geometricians, in introducing them to a ready acquaintance with the terms and principles of this difficult science, and preparing them for making further advances in mathematical knowledge.

THE

THE AUTHOR's

PREFACE.

I HAVE neither vanity, nor humility enough to publifh thefe Elements for the fake of obtaining the name of Author: in this undertaking, utility is my fole object. Nor indeed could any inferior motive have been fufficient, to fupport me under the tedious labours, to which my defign has obliged me to fubmit. What I here offer to the Public, is the refult of numberlefs efforts, obfervations, and corrections, continued through twenty years. Though profeffionally engaged in explaining the fublimer parts of modern Geometry in one of the firft Univerfities in Europe, I have not thought it beneath me to dedicate my leifure hours to a work of this kind: for, the farther I have carried my refearches, the more fully I have been convinced, of what all who are not wholly ftrangers to the fubject muft perceive, that well-written Elements of Geometry were ftill among the *defiderata* of fcience.

If thefe Elements fhould fail of obtaining applaufe on the firft curfory perufal, I fhall not be difappointed. The fuperficial reader will

eafily perfuade himfelf, that they are only a fervile repetition of what was faid two thoufand years ago by the great Father of Geometry. Were this the cafe, I fhould at leaft have the merit of having convinced myfelf by long experience, of the preference due to the method of Euclid above any other which has been fubftituted in its room. But good judges, who know how to difcriminate more accurately, will perhaps be of a different opinion. They will be aware of the rocks that were to be fhunned, and the difficulties that were to be furmounted, in executing this defign; and will eafily perceive, that without indulging an affectation of novelty, I have often been under the neceffity of ftriking out into unbeaten paths; by which means obftacles have fometimes perhaps been avoided, which had before frequently embarraffed both the Pupil and the Teacher.

All the Elementary Propofitions, which can be of any ufe in a complete courfe of Mathematics, are here reduced to about an hundred. To make the proper felection required judgment, fkill, and attention. How far I have fucceeded in the attempt, muft be left to the Public to determine; and the fuccefs of my work will be the teft of their approbation.

ELE-

ELEMENTS

OF

GEOMETRY.

DEFINITIONS.

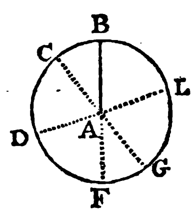
1. A *Solid* is that which hath length, breadth and thickneſs. A book, for inſtance, is a ſolid, ſince it is long, broad and thick.

2. A *Surface* is that which has length and breadth, but no thickneſs. A leaf of fine paper may repreſent a ſurface.

3. A *Line* is that which has length, but neither breadth nor thickneſs. A hair may repreſent a line; a thread drawn ſtraight, a right line.

4. A *Point* has neither length, breadth, nor thickneſs. A fine grain of ſand may repreſent a point.

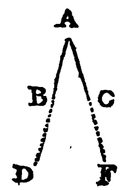
5. If a line AB is carried about a point A, in ſuch manner, that its other extremity paſſes from B to C, from C to D, &c. the point B, in its revolution, will deſcribe a curve line, BCDFGLB. This curve line is called the *Circumference* of a circle. The *Circle* is the ſpace incloſed by this circumference. The point A, which in the formation of the circle is at reſt, is called the *Center*. The right lines AC, AD, AF, &c. drawn from the center to the circumference, are called *Radii*. A *Diameter* is a right line which paſſes through the center, and is terminated both ways by the circumference: the line DAL, for example, is a diameter. An *Arc* is a part of the circumference, as FG.



6. The circumference of a circle is divided into 360 equal parts, which are called *Degrees*; every degree is divided into 60 parts, which are called *Minutes*; every minute is divided into 60 parts, which are called *Seconds*. A large circumference has the ſame number of degrees with a ſmall one, but the degrees in the former are greater than thoſe in the latter.

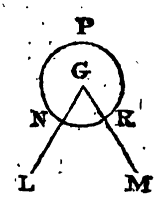
7. Two

7. Two right lines which are drawn from the same point, and recede from each other, form an opening which is called an *Angle*. An angle is commonly expressed by three letters; and it is usual to place in the middle the letter which marks the vertical point of the angle; thus we say, the angle BAC, and not the angle ABC, or ACB.



8. The magnitude of an angle does not depend upon the length of the lines which form it, but upon their distance from each other. How far soever the lines AB, AC are extended, the angle remains the same. One angle is greater than another, when the lines which form it are more distant. The angle BAL (see fig. def. 5.) is greater than the angle CAB, because the lines AB, AL, are more distant from each other than the lines AC, AB. If the legs of a pair of compasses be a little separated, an angle is formed; if they be opened wider, the angle becomes greater; if they be brought nearer, the angle becomes less.

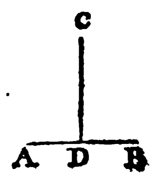
9. If a point of the compasses be applied to the point G, and a circumference NRP be described, the arc NR, contained within the two lines GL, GM, will measure the magnitude of the angle LGM. If the arc NR, for example, is an arc of 40 degrees, the angle LGM is an angle of 40 degrees.



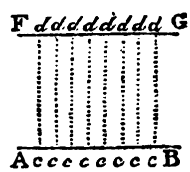
10. There

10. There are three kinds of angles; a *Right* angle, which is an angle of 90 degrees; an *Obtuse* angle, which contains more than 90 degrees; and an *Acute* angle, which contains less than 90 degrees.

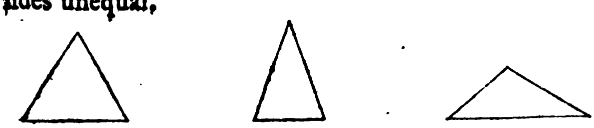
11. One line is *Perpendicular* to another line, when the two angles which it makes with that other line are equal: thus, the line CD is perpendicular to the line AB, if the angles CDA, CDB contain an equal number of degrees.



12. Two lines are *Parallel*, when all perpendiculars drawn from one to the other are equal: thus, the lines FG, AB are parallel, if all the perpendiculars, *c*, *d*, &c. are equal.

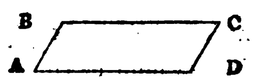


13. A *Triangle* is a surface enclosed by three right lines called its *Sides*. An *equilateral* triangle is that which has the three sides equal: an *isosceles* triangle has two of its sides equal: a *scalene* triangle has its three sides unequal.

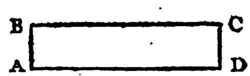


14. A *Quadrilateral* figure is a surface inclosed by four right lines, which are called its sides.

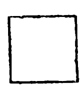
15. A *Parallelogram* is a quadrilateral figure, which has its opposite sides parallel; thus, if the side BC is parallel to the side AD, and the side AB to the side DC, the quadrilateral figure ABCD is called a parallelogram.



16. A *Rectangle* is a quadrilateral figure, all the angles of which are right angles as ABCD.



17. A *Square* is a quadrilateral figure, the sides of which are all equal, and all its angles right angles.

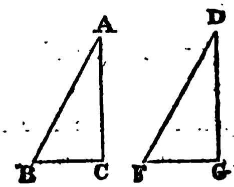


18. A *Trapezium* is any quadrilateral figure not a parallelogram.

19. Those figures are *Equal*, which inclose an equal space: thus, a circle and a triangle are equal, if the space included within the circumference of the circle be equal to that contained in the triangle.

20. Those

20. Thofe figures are *Identical*, which are equal in all their parts, that is, which have their angles equal, and their fides equal, and inclofe equal fpaces, as BAC, FDG.



It is manifeft that two figures are identical, which, being placed one upon the other, perfectly coincide; for, in that cafe, they muft be equal in all their parts.

N. B. A line, ufed fimply, always denotes a right line.

AXIOM. Two right lines cannot entirely inclofe a fpace: this requires at leaft three lines.

OF

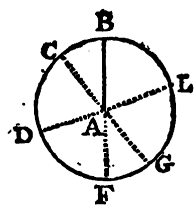
RIGHT LINES

AND

RECTILINEAL FIGURES.

PROPOSITION I.

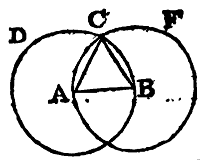
THE radii of the ſame circle are all equal.



The revolution of the line AB about the point A being neceſſary to form the circle BCDFGLB (def. 5.) when, in revolving, the point B is upon the point C, the whole line AB muſt be upon the line AC; otherwiſe two right lines would incloſe a ſpace, which is impoſſible: whence the radius AC is equal to the radius AB. In like manner it may be proved, that the radii AD, AF, AG, &c. are equal to AB: they are therefore equal among themſelves.

PROP. II.

To describe an equilateral triangle upon a given line.



Let AB be the given line, upon which it is required to describe a triangle, the three sides of which are equal.

From the point A, with the radius AB, describe the circumference BCD; from the point B, with the radius BA, describe the circumference ACF; and from the point C, where these two circumferences cut each other, draw the two right lines CA, CB: ACB is an equilateral triangle,

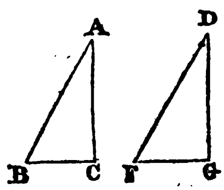
For, the line AC is equal to the line AB,* 1 because these two lines are radii of the same circle BCD: And, the line BC is equal to the line AB because 1 these two lines are radii of the same circle ACF. Whence, the lines AC and BC, being each equal to the same line AB, are equal to one another; and all the three sides of the triangle ACB are equal; that is, the triangle is equilateral.

* The figures refer to the preceding propositions.

PROP.

PROP. III.

Triangles which have two ſides and the angle contained by them equal, are identical.

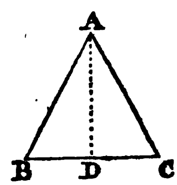


In the two triangles BAC, FDG, if the ſide DF be equal to the ſide AB, and the ſide DG equal to the ſide AC, and alſo the angle D equal to the angle A, the two triangles are identical.

Suppoſe the triangle FDG placed upon the triangle BAC, in ſuch manner that the ſide DF falls exactly upon the ſide equal to it, AB. Since the angle D is equal to the angle A, the ſide DG muſt fall upon the ſide equal to it, AC: alſo, the point F will be upon the point B, and the point G upon the point C: conſequently, the line FG muſt fall wholly upon the line BC; otherwiſe two right lines would incloſe a ſpace, which is impoſſible. The three ſides of the triangle FDG, therefore, coincide in all points, with the three ſides of the triangle BAC; and the two triangles have their ſides and angles equal, and incloſe an equal ſpace, that is, (def. 20.) they are identical.

PROP. IV.

In an isosceles triangle the angles at the base are equal.



Let the triangle BAC have its sides AB, AC, equal; the angles B and C, at the base, are also equal.

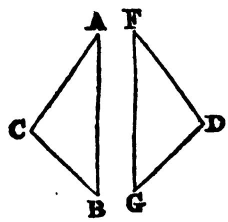
Conceive the angle A to be bisected by the right line AD.

In the triangles BAD, DAC, the sides AB, AC are, by supposition, equal; the side AD is common to the two triangles; and the angles at A are supposed equal. These two triangles have then the two sides, and the angle contained by them, equal: they are therefore 3 identical, or have all their parts equal: whence the angles B and C are equal.

PROP.

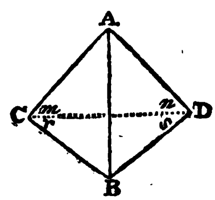
PROP. V.

Triangles which have their three sides equal, are identical.



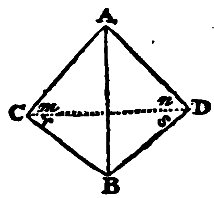
In the two triangles ACB, FDG, let the side AC be equal to the side FD, the side CB equal to the side DG, and the side AB to the side FG; these two triangles are identical.

Let the two triangles be so joined, that the side FG shall coincide with the side AB, and draw the right line CD.



Since, in the triangle CAD, the side AC is equal to the side AD, the triangle is isosceles; whence (def. 13.) the angles at the base, *m* and *n*, are equal.

 Since,



Since, in the triangle CBD, the ſide BC is equal to the ſide BD, the triangle is iſoſceles; whence (def. 13.) the angles at the baſe, *r* and *s*, are equal.

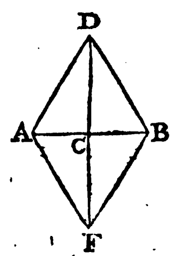
And becauſe the angle *m* is equal to the angle *n*, and the angle *r* equal to the angle *s*, the whole angle C is equal to the whole angle D.

Laſtly, becauſe in the two triangles ACB, ADB, the ſide AC is equal to the ſide AD, and the ſide CB equal to the ſide DB, and alſo the angle C equal to the angle D, theſe two triangles have two ſides and the contained angle equal, and are therefore 3 identical.

PROP.

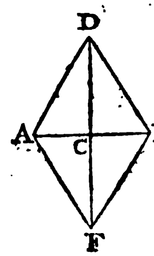
PROP. VI.

To divide a right line into two equal parts.



Let the right line, which it is required to divide into two equal parts, be AB. Upon AB draw [a] the equilateral triangle ADB; and on the other side of the same line AB draw the equilateral triangle AFB; draw also the right line DF; AC is equal to CB.

In the two larger triangles DAF, DBF, the sides DA, DB, are equal, because they are sides of an equilateral triangle; the sides AF, BF, are equal for the same reason; and the side DF is common to the two triangles. These two triangles have then their three sides equal, and consequently [5] are identical, or have all their parts equal; whence the two angles at D are equal.

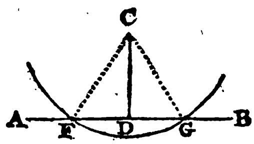


Again, in the two smaller triangles ADC, CDB, the side DA is made equal to the side DB; and the side DC is common to the two triangles: also, the two angles at D are equal. These two triangles have then two sides, and the contained angle, equal; they are therefore 3 identical, and AC is equal to CB, that is, AB is bisected.

PROP.

PROP. VII.

From a given point, out of a right line, to draw a perpendicular to that line.



Let C be the point from which it is required to draw a perpendicular to the right line AB.

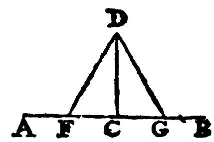
From the point C describe an arc of a circle which shall cut the line AB in two points, F and G. Then bisect the line FG, and to D, the point of division, draw the line CD: this line is perpendicular to the line AB. Draw the lines CF, CG.

In the triangles FCD, DCG, the sides, CF, CG are equal, [1] because they are radii of the same circle; the sides FD, DG are equal, because FG is bisected; and the side CD is common. These two triangles then, having the three sides equal, are [5] identical. Whence (def. 20.) the angle CDA is equal to the angle CDB, and consequently (def. 11.) the line CD is perpendicular to the line AB.

PROP.

PROP. VIII.

From a given point in a right line, to raiſe a perpendicular upon that line.



From the point C let it be required to raiſe a perpendicular upon the right line AB.

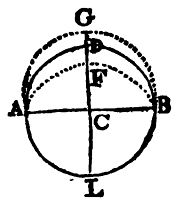
In AB take, at pleaſure, CF equal to CG; upon the line FG deſcribe an equilateral triangle FDG, and draw the line CD; this line will be perpendicular to AB.

In the triangles FDC, CDG, the ſides DF, DG are equal, becauſe they are ſides of an equilateral triangle; the ſides FC, CG are equal by conſtruction; and the ſide DC is common. Theſe two triangles, then, having the three ſides equal, are 5 identical. Therefore (def. 20.) the angle DCA is equal to the angle DCB, and conſequently (def. 11.) the line CD is perpendicular to the line AB.

PROP.

PROP. IX.

The diameter of a circle divides the circumference into two equal parts.



Let ADBLA be a circle; the diameter ACB bisects the circumference, that is, the arc ALB is equal to the arc ADB.

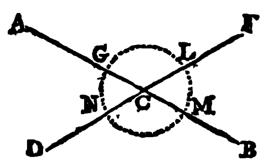
Conceive the circle to be divided, and the lower segment ACBLA to be placed upon the upper ACBDA; all the points of the arc ALB will fall exactly upon the arc ADB, and consequently these two arcs will be equal.

For, if the point L, for instance, does not fall upon the arc ADB, it must fall either above this arc, as at G, or below it, as at F. If it falls on G, the radius CL will be greater than the radius CD; if it falls on F, the radius CL will be less than the radius CD; which [1] is impossible. The point L must then fall upon the arc ADB. In like manner, it may be proved, that all the other points of the arc ALB must fall upon the arc ADB: these two arcs are therefore equal.

PROP.

PROP. X.

A right line which meets another right line, forms with it two angles which are together equal to two right angles.



The line AC meeting the line DF, and forming with it the two angles ACD, ACF, these two angles are together equal to two right angles.

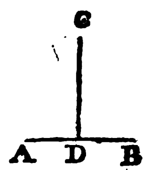
From the point C as a center describe, at pleasure, a circumference NGLMN.

The line NCL, being a diameter, divides the circumference 9 into two equal parts. The arc NGL is therefore half the circumference, which contains (def. 6.) 180, or twice 90, degrees. Therefore the angles ACD, ACF, which taken together are measured by the arc NGL, are twice 90 degrees, that is (def. 10.) are equal to two right angles.

PROP.

PROP. XI.

A right line drawn perpendicularly to another right line, makes right angles with it.

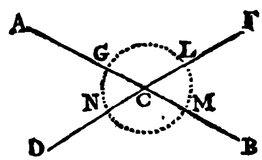


If the line CD be perpendicular to the line AB, the angle CDA is a right angle, and alſo the angle CDB.

For the line CD, meeting the line AB, forms with it two angles which are together [10] equal to two right angles; and theſe two angles are equal, becauſe CD is perpendicular to AB: wherefore each angle is a right angle.

PROP. XII.

If two lines cut each other, the vertical or oppofite angles are equal.



Let the lines AB, DF, cut each other at the point C; the angles ACD, FCB, which are called vertical or oppofite angles, are equal.

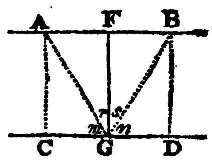
From the point C as a center defcribe, at pleafure, a circumference NGLMN.

Since the line NCL is a diameter, the arc NGL is 9 half the circumference: alfo fince GCM is a diameter, the arc GLM is 9 half the circumference: therefore the arcs NGL, GLM are equal. From thefe two arcs take away the common part GL, there will remain the arc NG equal to the arc LM: confequently, the angles ACD, FCB, which are meafured by thefe two arcs, are alfo equal.

PROP.

PROP. XIII.

If a line is perpendicular to one of two parallel lines, it is also perpendicular to the other.

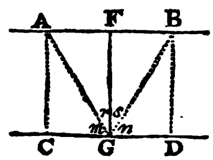


Let AB, CD be two parallel lines: if the line FG makes right angles with CD, it will also make right angles with AB.

Take at pleasure GC equal to GD; at the points C and D, raise the perpendiculars, CA, DB; and draw the lines GA, GB.

In the two triangles ACG, BDG, because the line AB is parallel to the line CD, the perpendiculars CA, DB are necessarily equal, as appears from the definition of parallel lines (def. 12.): the lines CG, DG, are equal by construction; and the angles C and D are right angles. The two triangles ACG, BDG have then two sides, and the contained angle, equal: they are therefore

therefore [3] identical. Whence the side GA is equal to the side GB, and the angle *m* equal to the angle *n*.

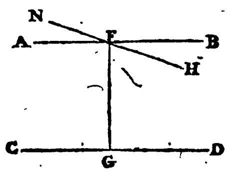


Again, in the triangles AGF, FGB, the side GA is equal to the side GB, as hath been proved; and the side GF is common. Moreover, the angle *r* is equal to the angle *s*: for, if from the two right angles FGC, FGD, be taken away the equal angles *m* and *n*, there will remain the equal angles *r* and *s*. The triangles AGF, FGB have then two sides, and the contained angle, equal: they are therefore [3] identical. Wherefore the angles GFA, GFB are equal, and consequently are right angles.

PROP.

PROP. XIV.

If one line is perpendicular to two other lines, these two lines are parallel.

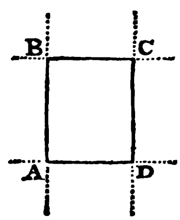


Let the line FG make right angles with the lines AB and CD; these two lines are parallel.

If the line AB be not parallel to the line CD, another line, as NH, may be drawn through the point F parallel to the line CD. But this is impossible; for if the line NH were parallel to the line CD, the line FG, making right angles with CD, would also [13] make right angles with NH; which cannot be, because, by supposition, it makes right angles with AB.

PROP. XV.

The opposite sides of a rectangle are parallel.



In the rectangle ABCD, the side BC is parallel to the side AD, and the side AB parallel to the side DC.

Produce each of the sides both ways.

The line AB is perpendicular to the two lines BC, AD; the two lines BC, AD are therefore [14] parallel. In like manner, the line AD is perpendicular to the two lines AB, DC; the two lines AB, DC, are therefore [14] parallel.

PROP.

PROP. XVI.

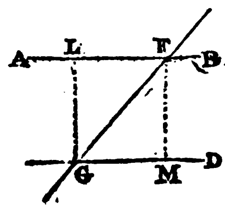
The oppofite fides of a rectangle are equal.

In the rectangle ABCD, [fee fig. to the preceding propofition] the fide AB is equal to the fide DC, and the fide BC equal to the fide AD.

For, fince the fide BC is parallel to the fide AD, the perpendiculars AB, DC are (def. 12.) equal; and, fince the fide AB is parallel to the fide DC, the perpendiculars BC, AD are equal.

PROP. XVII.

A right line, falling upon parallel lines, makes the alternate angles equal.



Let the line FG cut the parallels AB, GD; the angles AFG, FGD, which are called *alternate angles*, are equal.

From the point G draw GL perpendicular to the line AB; and from the point F draw FM perpendicular to the line GD.

Since the line GL is perpendicular to AB, it is alſo 13 perpendicular to the parallel line GD. In like manner, ſince the line FM is perpendicular to the line GD, it is alſo 13 perpendicular to the parallel line AB. Whence the quadrilateral figure GLFM is a rectangle, its four angles being right angles.

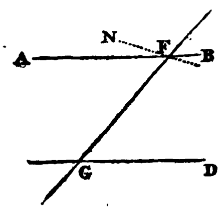
In the triangles GLF, FMG, the ſides LF, GM are equal, becauſe they are oppoſite ſides of the ſame rectangle:

rectangle: the ſides LG, FM, are equal for the ſame reaſon; and the ſide FG is common. The two triangles GLF, FMG, have then the three ſides equal, and conſequently 5 are identical. Wherefore the angle LFG, oppoſite to the ſide LG, is equal to the angle FGM, oppoſite to the ſide FM.

Remark. In identical triangles, the equal angles are always oppoſite to equal ſides, as appears in this propoſition.

PROP. XVIII.

If one right line, falling upon two others, makes the alternate angles equal, these two lines are parallel.

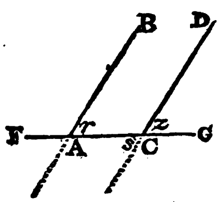


Let the alternate angles AFG, FGD, be equal, the lines AB, GD are parallel.

If the line AB is not parallel to the line GD, another line, as NF, may be drawn through the point F parallel to GD. But this is impossible: for, if the line NF were parallel to the line GD, the angle FGD would be [17] equal to the angle NFG, since these two angles would be alternate angles between two parallel lines; which cannot be, because, by supposition, the angle FGD is equal to the angle AFG.

PROP. XIX.

If one right line falls upon two parallel right lines, it makes the interior angle equal to the exterior.



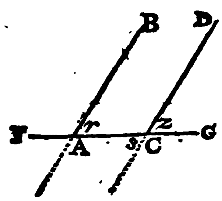
Let the line FG meet the parallel lines BA, DC, the interior angle *r* is equal to the exterior angle *z*.

Produce the lines BA, DC.

The angle *r* is [17] equal to the angle *s*, because these are alternate angles, made by a right line falling upon two parallel lines: and the angles *s* and *z* are [12] equal, because they are vertical or opposite angles: therefore the angle *r* is equal to the angle *z*.

PROP. XX.

If one right line, falling upon two other right lines, makes the internal angle equal to the external, these two lines are parallel.



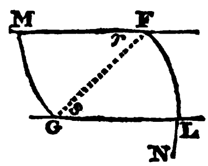
Let the internal angle r be equal to the external angle z, the lines BA, DC are parallel.

The angle r is equal to the angle z by supposition; and the angle z is [12] equal to the angle s, because they are opposite angles. The alternate angles r, s, are therefore equal, and consequently [18] the lines BA, DC are parallel.

PROP.

PROP. XXI.

Through a given point to draw a line parallel to a given line.



Let G be the point through which it is required to draw a line parallel to the given line MF.

From any point G describe, at pleasure, the arc FN: from the point F, in which the arc FN cuts the line MF, with the distance GF, describe the arc GM meeting the line MF in M: then make FL equal to GM, and draw the line GL; this line is parallel to the line MF.

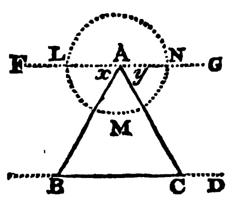
Draw the line GF.

The arcs GM, FL are equal by construction; therefore the alternate angles *r*, *s*, which are measured by these arcs (def. 9.) are equal: and consequently [18] the lines GL MF are parallel.

PROP.

PROP. XXII.

The three angles of a triangle are equal to two right angles.



In the triangle BAC, the three angles B, A, C, are together equal to two right angles.

Produce the ſide BC both ways; through the point A, draw a line FG parallel to BC; and from the point A, as a center, deſcribe any circumference LMN.

The angle B is [17] equal to the angle x, becauſe theſe are alternate angles made by a right line falling upon two parallel lines. For the ſame reaſon the angle C is equal to the angle y.

Becauſe LAN is a diameter, the arc LMN is half the circumference: therefore the three angles x, A, y, which are meaſured by this arc, are together equal to two right angles.

But

But the angle x is equal to the alternate angle B, and the angle y to the alternate angle C.

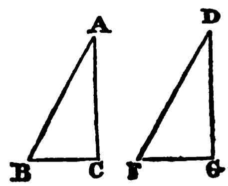
Therefore, subſtituting B for x, and C for y, the three angles B, A, C, are together equal to two right angles.

COROLLARY. Hence, if two angles of any triangle be known, the third is alſo found; ſince the third angle is that which the other two, taken together, want of two right angles.

PROP.

PROP. XXV.

Triangles which have two angles, and the ſide which lies between them, equal, are identical.



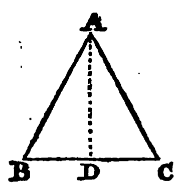
In the two triangles BAC, FDG, if the angle F is equal to the angle B, the angle G equal to the angle C, and the ſide FG equal to the ſide BC, theſe two triangles are identical.

Conceive the triangle FDG placed upon the triangle BAC, in ſuch manner that the ſide FG ſhall fall exactly upon the equal ſide BC. Since the angle F is equal to the angle B, the ſide FD muſt fall upon the ſide BA: and ſince the angle G is equal to the angle C, the ſide GD muſt fall upon the ſide CA. Thus the three ſides of the triangle FDG will be exactly placed upon the three ſides of the triangle BAC; and conſequently the two triangles 5 are identical.

PROP.

PROP. XXVI.

If two angles of a triangle are equal, the ſides oppoſite to theſe angles are alſo equal.



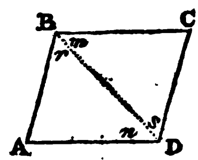
Conceive the angle A to be biſected by the line AD.

In the triangles BAD, DAC, the angle B is equal to the angle C, by ſuppoſition; and the angles at A are alſo equal. Theſe two triangles have then two angles equal; the third angle will therefore [23] be equal: whence the angles at D are equal. Moreover, the ſide AD is common to the two triangles. Theſe two triangles therefore, having two angles, and the ſide which lies between them equal, are [25] identical. Whence the ſide AB is equal to the ſide AC.

PROP.

PROP. XXVII.

The oppoſite ſides of a parallelogram are equal.



In the parallelogram ABCD, the ſide AB is equal to the ſide DC, and the ſide BC equal to the ſide AD.

Draw the line BD which is called the *Diagonal.*

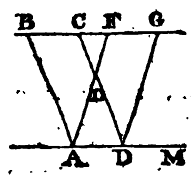
Becauſe BC is parallel to AD, the alternate angles *m* and *n* are equal. In like manner, becauſe AB is parallel to DC, the alternate angles *r* and *s* are equal. Alſo, the ſide BD is common to the two triangles BAD, BCD. Theſe two triangles have then two angles, and the ſide which lies between them, equal, and are therefore 3 identical. Wherefore the ſide AB, oppoſite to the angle *n*, is 26 equal to the ſide DC, oppoſite to the equal angle *m*; and the ſide BC, oppoſite to the angle *s*, is equal to the ſide AD, oppoſite to the equal angle *r*.

COR. Hence it follows, that the diagonal biſects the parallelogram: for, the triangles BAD, BCD, having the three ſides equal, are identical.

PROP.

PROP. XXVIII.

Parallelograms which are between the ſame parallels, and have the ſame baſe, are equal.

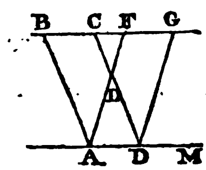


Let the two parallelograms ABCD, AFGD, be between the ſame parallels BG, AM, and upon the ſame baſe AD; the ſpace encloſed within the parallelogram ABCD, is equal to the ſpace encloſed within the parallelogram AFGD.

In the two triangles BAF, CDG, the ſide BA of the former triangle is equal to the ſide CD of the latter, becauſe they are oppoſite ſides of the ſame parallelogram. For the ſame reaſon, the ſide FA is equal to the ſide GD. Moreover, BC is equal to AD, becauſe they are oppoſite ſides of the ſame parallelogram. For the ſame reaſon, AD is equal to FG. BC is therefore equal to FG. If to both theſe CF be added, BF will be equal to CG. Whence

 the

the two triangles BAF, CDG, having the three ſides equal 5 are identical, and conſequently have equal ſurfaces.

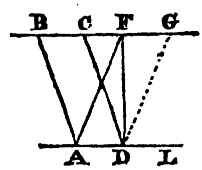


If from theſe two equal ſurfaces be taken the ſmall triangle CLF, which is common, there will remain the trapezium ABCL equal to the trapezium LFGD. To theſe two trapezia, add the triangle ALD, and the parallelogram ABCD will be equal to the parallelogram AFGD.

PROP.

PROP. XXIX.

If a triangle and a parallelogram are upon the same base, and between the same parallels, the triangle is equal to half the parallelogram.



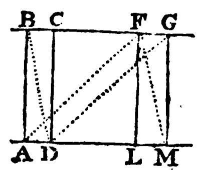
Let the parallelogram ABCD, and the triangle AFD, be upon the same base AD, and between the same parallels BG, AL; the triangle AFD is half the parallelogram ABCD.

Draw DG parallel to AF.

Because the parallelogram AFGD is bisected by the diagonal FD, (prop. 27. cor.) the triangle AFD is half the parallelogram AFGD. But the parallelogram AFGD is equal to the parallelogram ABCD, because these two parallelograms are upon the same base and between the same parallels: therefore the triangle AFD is equal to half the parallelogram ABCD.

PROP. XXX.

Parallelograms which are between the ſame parallels, and have equal baſes, are equal.



Let the two parallelograms ABCD, LFGM be between the ſame parallels BG, AM, and have the equal baſes AD, LM; theſe two parallelograms are equal.

Draw the lines AF, DG.

Becauſe AD is equal to LM, and LM to FG, AD is equal to FG; and they are parallel by conſtruction: alſo AF and DG are parallel; for, if DG be not parallel to AF, another line may be drawn parallel to it; whence FG will become greater or leſs than AD; which is impoſſible, becauſe FG has been proved to be equal to AD. AF and AG are therefore parallel, and AFGD a parallelogram.

Now

Now the parallelogram ABCD is [28] equal to the parallelogram AFGD, becauſe theſe two parallelograms are between the ſame parallels, and have the ſame baſe AD. And the parallelogram AFGD is equal to the parallelogram LFGM, becauſe theſe two parallelograms are between the ſame parallels, and have the ſame baſe FG. The parallelogram ABCD is therefore equal to the parallelogram LFGM,

PROP. XXXI.

Triangles which are between the ſame parallels and have equal baſes, are equal.

Let the two triangles ABD, LFM, [ſee fig. to the preceding propoſition] be between the ſame parallels BG, AM, and upon the equal baſes AD, LM, theſe two triangles are equal.

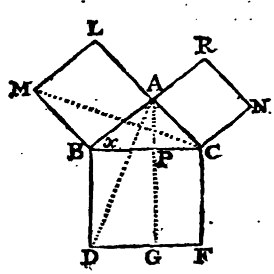
Draw DC parallel to AB, and MG parallel to LF.

The two parallelograms ABCD, LFGM, are equal, [30] becauſe they are between the ſame parallels, and have equal baſes. But the triangle ABD is [29] one half of the parallelogram ABCD; and the triangle LFM is one half of the parallelogram LFGM. Theſe two triangles are therefore equal.

PROP.

PROP. XXXII.

In a right-angled triangle the ſquare of the hypothenuſe, or ſide ſubtending the right angle, is equal to the ſquares of the ſides which contain the right angle.



In the triangle BAC, let the angle A be a right angle. Upon the hypothenuſe BC deſcribe the ſquare BDFC; upon the ſide AB deſcribe the ſquare ALMB, and upon the ſide AC, the ſquare ARNC; the ſquare BDFC is equal to the two ſquares ALMB, ARNC taken together.

Draw the right lines MC, AD; and draw AG parallel to BD.

Becauſe

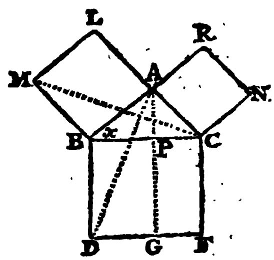
Becaufe the fquare or parallelogram MLAB, and the triangle MCB, are between the fame parallels LC, MB, and have the fame bafe MB, the triangle MCB is [29] equal to half the fquare ALMB.

Again, becaufe the rectangle or parallelogram DGPB, and the triangle DAB are between the fame parallels GA, DB, and have the fame bafe DB, the triangle DAB is [29] equal to half the rectangle DGBP.

Further, fince the fide MB of the triangle MBC and the fide AB of the triangle ABD, are fides of the fame fquare, they are (def. 17.) equal. Alfo, fince the fide BC of the firft triangle, and the fide BD of the fecond triangle are fides of the fame fquare, they are equal. And becaufe the angle MBC of the firft triangle is compofed of a right angle and the angle x, and the angle ABD of the fecond triangle is compofed of a right angle, and the fame angle x, therefore thefe two angles, contained between the equal fides, MB, BC, and AB, BD, are equal. Wherefore, the two triangles MBC, ABD, having two fides and the contained angle equal, are [3] identical, and confequently equal.

But the triangle MBC is half the fquare MLAB; and the triangle ABD is half the rectangle BDGP. The fquare and the rectangle are therefore equal.

In



In the ſame manner it may be demonſtrated, that the ſquare ARNC and the rectangle CFGP are equal: whence it follows, that the whole ſquare BDFC is equal to the two ſquares MLAB, ARNC, taken together.

OF

CIRCLES.

DEFINITIONS.

1. A Right line (see fig. prop. xxxiii. AB) terminated both ways by the circumference of a circle, is called a *Chord*.

2. A line (see fig. prop. xxxix. AB) which meets the circumference in one point only, is called a *Tangent*; and the point T is called the *Point of Contact*.

3. An angle (see fig. prop. xxxiii. ABD) which has its vertex in the circumference of a circle, is called an *Angle in the Circle*.

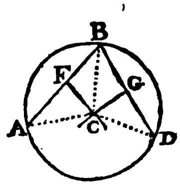
4. A part of a circle confined between two radii, (see fig. prop. xxxiv. ACBFA) is called a *Sector*.

5. A part of a circle (see fig. prop. xxxv. AGBDA) terminated by a chord, is called a *Segment of a Circle*.

E PROP.

PROP. XXXIII.

To draw the circumference of a circle through three given points.



Let there be three given points, A, B, D, through which it is required to draw the circumference of a circle.

Draw the right lines AB, BD, and biſect them: from the points of diviſion, F, G, raiſe the perpendiculars FC, GC; and at the point C, with the radius CA, deſcribe the circumference of a circle: this circumference will paſs through the points B and D. Draw the lines CA, CB, CD.

In the triangles CFA, CFB, the ſide FA is equal to the ſide FB by conſtruction; the ſide FC is common; and the two angles at F are right angles. Theſe two triangles have then two ſides, and the angle contained

tained by them, equal: they are therefore [3] identical. Confequently the fide CB is equal to the fide CA.

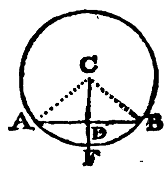
For the fame reafon, the triangles CGB, CGD are alfo identical. whence the fide CD is equal to the fide CB, and confequently equal to CA.

And fince the right lines CB, CD are equal to the right line CA, it is manifeft [1] that the circumference which paffes through the point A, muft alfo pafs through the point D.

PROP.

PROP. XXXIV.

If a radius bifects a chord, it is perpendicular to that chord.



If the radius CF bifects the chord AB, the angles CDA, CDB are right angles. Draw the radii CA, CB.

In the triangles CDA, CDB, the fides CA, CB, being radii, are [1] equal; the fides AD, DB, are equal by fuppofition; and the fide CD is common. Thefe two triangles, having the three fides equal, are therefore [5] identical. Whence the angles CDA, CDB, are equal, and confequently [10] are right angles.

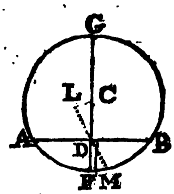
COR. The two angles at C are alfo [5] equal.

Hence it appears, that any angle ACB may be bifected, by defcribing from its vertex C as the center, with any radius AC, an arc AFB; bifecting the chord of that arc AB; and then drawing, from the point of divifion D, the right line CD: for it may then be fhewn as in the propofition, that the triangles ACD, DCB are identical, and confequently the angles at C equal.

PROP.

PROP. XXXV.

To find the center of a circle.

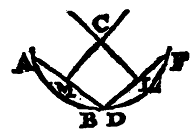


Let the circle of which it is required to find the center be AGBF. Draw any chord AB; bisect it, and from the point of division D, raise a perpendicular FG: this line will pass through the center, and consequently if it be bisected, the point of division will be the center.

If the center of the circle be not in the line FG, it must be somewhere out of it, for instance at the point L. But this is impossible; for if the point L were the center, the right line LM would be a radius; and since this line bisects the chord AB, it is 34 perpendicular to AB; which cannot be, since CD is perpendicular to AB.

PROP. XXXVI.

To find the center of an arc of a circle.



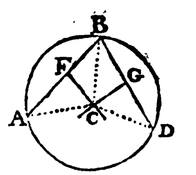
Let ABDF be the arc of which it is required to find the center. Draw any two chords, AB, DF; bisect them, and from the points of division, raise the perpendiculars MC, LC; the point C, in which these two perpendiculars cut each other, is the center of the arc.

For 35 the perpendicular MC, and the perpendicular LC, both pass through the center of the same circle; this center must therefore be the point C, which is the only point common to the two perpendiculars.

PROP.

PROP. XXXVII.

If three equal lines meet in the fame point within a circle, and are terminated by its circumference, they are radii of that circle.



The lines CA, CB, CD, drawn from the fame point C, within a circle, and terminated by it, being equal, the point C is the center of the circle. Draw the lines AB, BD; bifect them, and let the points of divifion be F, G; and draw the lines CF, CG.

In the triangles CFA, CFB, the fides CA, CB are equal by fuppofition; the fides FA, FB are equal by conftruction; and the fide CF is common. Thefe two triangles have then the three fides equal: they are therefore 5 identical. Whence the two angles at F are equal, and the line FC (def. 11.) is perpendicular to the chord AB. And fince this perpendicular

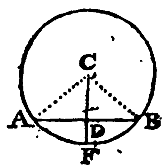
bisects the chord AB, it must 35 pass through the center of the circle. In like manner it may be demonstrated, that the line GC also passes through the center. Whence the point C is the center of the circle, and CA, CB, CD are radii.

PROP.

of circle
of circle

PROP. XXXVIII.

If the radius of a circle is perpendicular to a chord, the radius bisects both the chord and the arc of the chord.



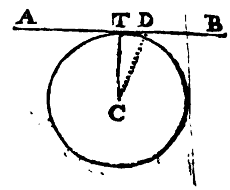
Let the radius CF be perpendicular to the chord AB; the right line AD is equal to the right line DB, and the arc AF equal to the arc FB. Draw the right lines CA, CB.

In the large triangle ACB, the side CA is [1] equal to the side CB, because they are radii of the same circle: the angle A is therefore equal to the angle B.

In the two smaller triangles CDA, CDB, the angle A is [4] equal to the angle B; the angles at D are right angles, and therefore equal; and the angles at C are consequently [23] equal. Also, the side CA is equal to the side CB, and the side CD is common. These two triangles having then two sides, and the angle contained by them, equal, are [3] identical; whence the side AD is equal to the side DB. Again, since the angles ACF, BCF are equal, the arcs AF, BF, which measure these angles, are also equal. The chord AB, and the arc AFB, are therefore bisected by the radius CF.

PROP. XXXIX.

A right line perpendicular to the extremity of a radius, is a tangent to the circle.



Let the line AB pass through the extremity of the radius CT, in such manner that the angles CTA, CTB shall be right angles: this line AB touches the circumference only in one point T. If AB touches the circumference in any other point, let it be D, and draw the line CD.

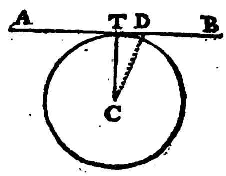
In the right-angled triangle CTD, the square of the hypothenuse CD is equal to the two squares of CT and TD taken together. The square of CD is therefore greater than the square of CT, and consequently the line CD is greater than the line CT, which is a radius. Therefore the point D is out of the circumference. And in like manner it may be shewn,

ſhewn, that every point in the line AB is out of the circumference except T; AB is therefore a tangent to the circle.

COR. Hence it follows, that a perpendicular is the ſhorteſt line which can be drawn from any point to a given line; ſince the perpendicular CT is ſhorter than any other line which can be drawn from the point C to the line AB.

PROP. XL.

If a right line be drawn touching a circumference, a radius drawn to the point of contact will be perpendicular to the tangent.



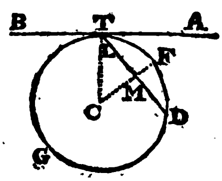
Let the line AB touch the circumference of a circle in a point T; the radius CT is perpendicular to the tangent AB.

For, all other lines drawn from the point C to the line AB, must paſs out of the circle to arrive at this line. The line CT is therefore the ſhorteſt which can be drawn from the point C to the line AB, and conſequently (39 Cor.) is perpendicular to the line AB.

PROP.

PROP. XLI.

The angle formed by a tangent and chord, is measured by half the arc of that chord.

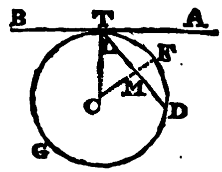


Let BTA be a tangent, and TD a chord drawn from the point of contact T; the angle ATD is measured by half the arc TFD; and the angle BTD is measured by half the arc TGD. Draw the radius CT to the point of contact, and the radius CF perpendicular to the chord TD.

The radius CF, being perpendicular to the chord TD, [38] bisects the arc TFD. TF is therefore half the arc TFD.

In the triangle CML, the angle M being a right angle, the two remaining angles are [22] equal to a right angle; whence the angle C is that which the angle L wants of a right angle. On the other side, since the radius CT is perpendicular to the tangent BA, the angle

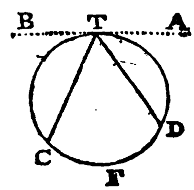
angle ATD is also that which the angle L wants of a right angle. The angle ATD is therefore equal to the angle C. But the angle C is measured by the arc TF; consequently, the angle ATD is also measured by the arc TF, which is half of TFD, the arc of the chord TD.



Again, the line TD forms with the line BA two angles ATD, BTD, which are [10] equal to two right angles, and consequently are measured by half the circumference. But the angle ATD is measured by half the arc TFD. The angle BTD must therefore be measured by half the arc TGD, since these two halves of arcs make up half the circumference.

PROP. XLII.

An angle at the circumference of a circle, is meaſured by half the arc by which it is ſubtended.



Let CTD be the angle at the circumference; it has for its meaſure, half the arc CFD by which it is ſubtended.

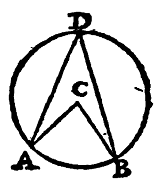
Suppoſe a tangent paſſing through the point T.

The three angles at T are meaſured by half the circumference (ſee prop. 22.) but the angle ATD is meaſured 41 by half the arc TD, and the angle BTC by half the arc TC; conſequently the angle CTD muſt be meaſured by half the arc CFD, ſince theſe three halves of arcs make up half the circumference.

PROP.

PROP. XLIII.

The angle at the center of a circle is double of the angle at the circumference.



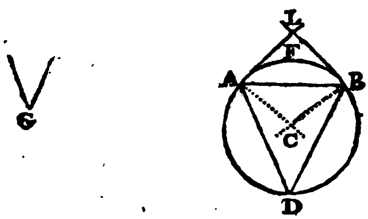
Let the angle at the circumference ADB, and the angle at the center ACB, be both ſubtended by the ſame arc AB; the angle ACB is double of the angle ADB.

For, the angle ACB is meaſured by the arc AB; and the angle ADB is [42] meaſured by half the ſame arc AB: the angle ACB is therefore double of the angle ADB.

PROP.

PROP. XLIV.

Upon a given line, to deſcribe a ſegment of a circle containing a given angle.



Let AB be the given line, and G the given angle: it is required to draw ſuch a circumference of a circle through the points A and B, that the angle D ſhall be equal to the angle G.

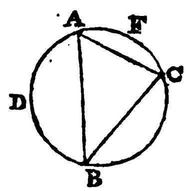
For this purpoſe, draw the lines AL, BL, in ſuch manner that the angles A and B ſhall be equal to the angle G; at the extremities of LA, LB, raiſe the perpendiculars AC, BC; and from the point C in which theſe two perpendiculars cut each other, with the radius CA or CB, deſcribe the circumference ADB; the angle D will be equal to the angle G.

The angle LAB, formed by the tangent AL and the chord AB is [41] meaſured by the half of the arc AFB; and the angle D at the circumference is alſo meaſured [42] by the half of the arc AFB: the angle D is therefore equal to the angle LAB. But the angle LAB is made equal to the angle G; the angle D is therefore equal to the angle G.

PROP.

PROP. XLV.

In every triangle, the greater side is opposite to the greater angle, and the greater angle to the greater side.



In the triangle ABC, if the side AB is greater than the side AC, the angle C opposite to the side AB, will be greater than the angle B opposite to the side AC. Draw the circumference of a circle through the three points A, C, B.

Since the chord AB is greater than the chord AC, it is manifest that the arc ADB is greater than the arc AFC; and consequently, the angle at the circumference C, which is measured 42 by half the arc ADB, is greater than the angle at the circumference B, which is measured by the half of the arc AFC.

Again,

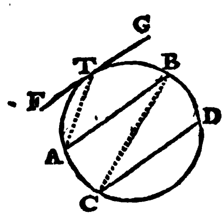
Again, if the angle C is greater than the angle B, the side AB, opposite to the angle C, will be greater than the side AC, opposite to the angle B.

The angle C is measured 42 by half the arc ADB, and the angle B by half the arc AFC. But the angle C is greater than the angle B: the arc ADB is therefore greater than the arc AFC; and consequently, the chord AB is greater than the chord AC.

PROP.

PROP. XLVI.

Two parallel chords intercept equal arcs.

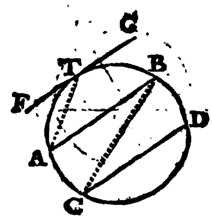


If the two chords AB, CD are parallel, the arcs AC, BD are equal. Draw the right line BC.

Becaufe the lines AB, CD are parallel, the alternate angles ABC, BCD are [17] equal. But the angle at the circumference ABC, is meafured [42] by half the arc AC; and the angle at the circumference BCD is meafured by half the arc BD: the arcs AC BD are therefore equal.

PROP. XLVII.

If a tangent and chord be parallel to each other, they intercept equal arcs.

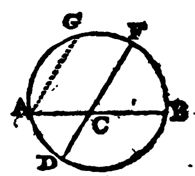


Let the tangent FG be parallel to the chord AB; the arc TA will be equal to the arc TB. Draw the right line TA.

Because the lines FG, AB are parallel, the alternate angles FTA, TAB are [17] equal. But the angle FTA, formed by a tangent and a chord, is measured [41] by half the arc TA; and the angle at the circumference TAB is measured [42] by half the arc TB. The halves of the arcs TA, TB, and consequently the arcs themselves, are therefore equal.

PROP. XLVIII.

The angle formed by the interfection of two chords, is meafured by half the two arcs intercepted by the two chords.



Let the two chords AB, DF cut each other at the point C; the angle FCB, or ACD, is meafured by half the two arcs FB, AD. Draw AG parallel to DF.

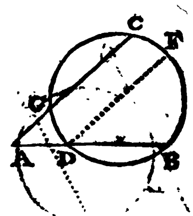
Becaufe the lines AG, DF are parallel, the interior and exterior angles GAB, FCB, are 19 equal. But the angle at the circumference GAB, is meafured by 42 half the arc GFB. The angle FCB is therefore alfo meafured by half the arc GFB.

Becaufe the chords AG, DF are parallel, the arcs, GF, AD are 46 equal: AD may therefore be fubftituted in the room of GF; whence the angle FCB is meafured by half the arcs AD, FB.

PROP.

PROP. XLIX.

The angle formed by two secants is measured by half the difference of the two intercepted arcs.



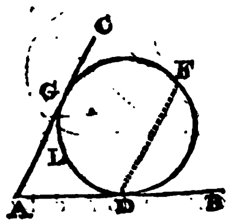
Let the angle CAB be formed by the two secants AC, AB, this angle is measured by half the difference of the two arcs GD, CB, intercepted by the two secants. Draw DF parallel to AC.

Because the lines AC, DF are parallel, the interior and exterior angles CAB, FDB are 19 equal. But the angle FDB is measured 42 by half the arc FB; the angle GAB is therefore also measured by half the arc FB.

Because the chords GC, DF are parallel, the arcs GD CF are 46 equal. The arc FB is therefore the difference of the arc GD and the arc CFB. Wherefore, the angle A has for its measure half the difference of the arcs GD, CFB.

PROP. L.

The angle formed by two tangents is measured by half the difference of the two intercepted arcs.

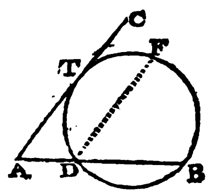


Let the angle CAB be formed by the two tangents AC, AB; this angle is measured by half the difference of the two arcs GLD, GFD. Draw DF parallel to AC.

Because the lines AC, DF are parallel, the interior and exterior angles CAB, FDB are 19 equal. But the angle FDB, formed by the tangent DB and the chord DF, is measured 41 by half the arc FD. Therefore the angle CAB is also measured by half the arc FD.

Because the tangent AC and the chord DF are parallel, the intercepted arcs GF, GD are 47 equal. The

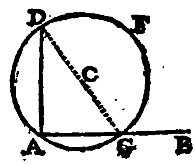
The arc FD is therefore the difference between the arc GLD and the arc GFD. Therefore the angle CAB, which is meaſured by half the arc FD, is alſo meaſured by half the difference of the arcs GLD, GFD.



COR. In the ſame manner it may be demonſtrated, that the angle formed by a tangent ATC, and a ſecant ADB, is meaſured by half the difference of the two intercepted arcs.

PROP. LI.

To raise a perpendicular at the extremity of a given line.



At the extremity A of the given line AB, let it be required to raise a perpendicular.

From any point C taken above the line AB, describe a circumference, passing through the point A, and cutting the line AB in any other point, as G. Draw the diameter GD, and the right line AD; this line AD will be perpendicular to the line AB.

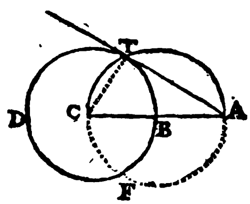
The angle DAG at the circumference is measured by [42] half the arc DFG, which is half the circumference, because DCG is a diameter. The angle DAG is therefore measured by one fourth part of the circumference, and consequently (def. 10.) is a right angle, whence the line AD is [11] perpendicular to the line AB.

COR. Hence it follows that the angle at the circumference which is subtended by a diameter, must be a right angle.

PROP.

PROP. LII.

From any point without a circle to draw a tangent to that circle.



From the point A let it be required to draw a tangent to the circle DTB.

Draw, from the center C, any right line CA; bisect this right line; and from the point of division B as a center describe the arc CTA. Lastly, from the point A, and through the point T, in which the two arcs cut each other, draw the right line AT; this right line AT will be a tangent to the circle DTB. Draw the radius CT.

The angle CTA at the circumference, being subtended by the diameter CA, is (51 cor.) a right angle; therefore the line TA is perpendicular to the extremity of the radius CT; and consequently it is a tangent to the circle DTB.

OF THE

MENSURATION

OF

SURFACES.

DEFINITIONS.

1. A *Mathematical Point* has neither length, breadth, nor thickneſs. The *Phyſical Point*, of which we are now to ſpeak, has a ſuppoſed length and breadth exceedingly ſmall.

2. A *Phyſical Line* is a ſeries of phyſical points: and conſequently, its breadth is equal to that of the phyſical points of which it is compoſed.

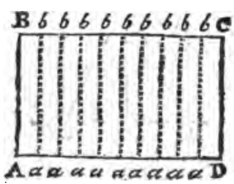
3. Since phyſical lines are compoſed of points, as numbers are compoſed of units, points may be called the *Units* of lines.

4. As, to multiply one number by another, is to take or repeat the firſt number as many times as there are units in the ſecond; ſo, to *multiply one line by another*, is to take or repeat the firſt line as many times as there are units, that is phyſical points, in the ſecond,

PROP.

PROP. LIII.

The ſurface of a rectangle is equal to the product of its two ſides.



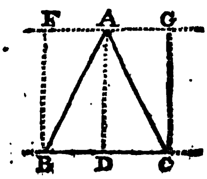
Let the rectangle be ABCD. If the phyſical line AB be multiplied by the phyſical line AD, the product will be the ſurface ABCD.

If as many phyſical lines, equal to AB, as there are phyſical points in the line AD, be raiſed perpendicularly upon AD, theſe lines, AB, *ab*, &c. will fill up the whole ſurface of the rectangle ABCD. Therefore the ſurface ABCD is equal to the line AB taken as many times as there are phyſical points in the line AD, that is, (def. 4.) equal to the line AB multiplied by the line AD.

PROP.

PROP. LIV.

The surface of a triangle is equal to half the product of its altitude and base.



If from the vertex of any angle, A, of the triangle BAC, be drawn AD perpendicular to the opposite side BC, this perpendicular is called the *height*, and the side BC the *base*, of the triangle. Now, the surface of the triangle is equal to half the product of the height AD and the base BC.

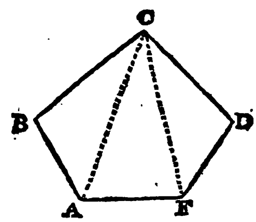
Produce BC both ways; through the point A, draw FG parallel to BC; and raise the two perpendiculars BF, CG.

Because the rectangle BFGC and the triangle BAC are between the same parallels and have the same base, the triangle is 29 half the rectangle. But the surface of the rectangle is equal 53 to the product of BF and BC. Therefore the surface of the triangle is equal to half the product of BF and BC, that is, of DA and BC.

PROP.

PROP. LV.

To meaſure the ſurface of any rectilineal figure.

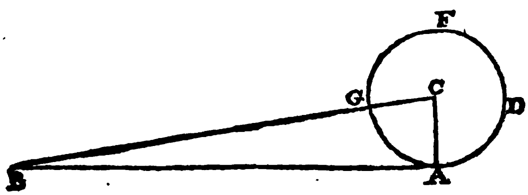


Let ABCDFA be the rectilineal figure, of which it is required to find the ſurface.

Divide the whole figure into triangles by drawing the lines CA, CF. Then drawing a perpendicular from the point B to the ſide CA, multiply theſe two lines : the half of their product will 54 give the ſurface of the triangle ABC. In the ſame manner let the ſurfaces of the remaining triangles ACF, FCD be found. Theſe three ſurfaces added together will give the whole ſurface of the figure ABCDFA.

PROP. LVII.

To draw a triangle equal to a given circle.



Let it be required to form a triangle, the ſurface of which ſhall be equal to that of the circle AGFDA.

At the extremity of any radius of the circle, CA, raiſe a perpendicular AB equal to the circumference AGFD, and draw the right line CB; the ſurface of the triangle BCA will be equal to that of the circle AGFDA.

The ſurface of the circle is equal [56] to half the product of the radius CA and the circumference or the line AB. The ſurface of the triangle is alſo equal [54] to half the product of its height CA, or radius, and its baſe BA, or the circumference. Therefore the ſurface of the triangle is equal to that of the circle.

OF

PROPORTION.

DEFINITIONS.

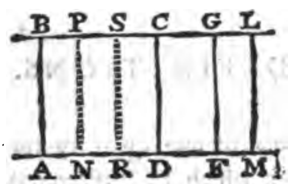
1. THE *Ratio* of one quantity to another is the number of times which the first contains the second: thus the ratio of 12 to 3 is four, because 12 contains 3 four times; "or more universally, *ratio* is the comparative magnitude of one quantity with respect to another."

2. Four quantities are *proportionals*, or *in geometrical proportion*, or "two quantities are said to have the same ratio with two others," when the first contains, or is contained in, the second, exactly the same number of times which the third contains, or is contained in, the fourth. Thus the four numbers, 6, 3, 8, 4 are proportionals, because 6 contains 3 as many times as 8 contains 4, and 3 is contained in six as many times as 4 is contained in 8, that is twice; which is thus expressed, 6 is to 3 as 8 to 4, or 3 is to 6 as 4 to 8.

PROP.

PROP. LVIII.

Parallelograms which are between the same parallels, are to one another as their bases.



Let the two parallelograms ABCD, FGLM, be between the same parallels BL, AM; the surface of the parallelogram ABCD, contains the surface of the parallelogram FGLM, as many times, exactly, as the base AD contains the base FM. Suppose, for example, that the base AD is triple of the base FM; in this case the surface ABCD will also be triple of the surface FGLM.

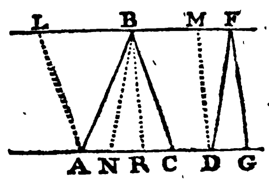
Divide the base AD into three parts, each of which is equal to the base FM; and draw from the points of division the lines NP, RS parallel to the side AB.

The

The parallelograms ABPN, FGLM, being between the fame parallels and having equal bafes, the parallelogram ABPN is 3° equal to the parallelogram FGLM. For the fame reafon, the parallelograms NPSR, RSCD are alfo equal to the parallelogram FGLM. The parallelogram ABCD is therefore compofed of three parallelograms, each of which is equal to the parallelogram FGLM. Confequently the parallelogram ABCD is triple of the parallelogram FGLM.

PROP. LIX.

Triangles which are between the ſame parallels are to one another as their baſes.



Let the two triangles ABC, DFG be between the ſame parallels LF, AG; the ſurface of the triangle ABC contains the ſurface of the triangle DFG as many times as the baſe AC contains the baſe DG. Suppoſe, for example, that the baſe AC is triple of the baſe DG; in this caſe the ſurface ABC will be triple of the ſurface DFG.

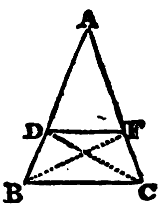
Divide the baſe AC into three equal parts AN, NR, RC, each of which is equal to the baſe DG; and draw the right lines BN, BR.

The triangles ABN, DFG, being between the ſame parallels, and having equal baſes, the triangle ABN is

is [31] equal to the triangle DFG. For the ſame reaſon the triangles NBR, RBC, are each equal to the triangle DFG. The triangle ABC is therefore compoſed of three triangles, each of which is equal to the triangle DFG. Therefore the triangle ABC is triple of the triangle DFG.

PROP. LX.

If a line be drawn in a triangle parallel to one of its ſides, it will cut the other two ſides proportionally.



In the triangle BAC, if the line DF is parallel to the ſide BC, it will cut the other two ſides in ſuch manner, that the ſegment AD will be to the ſegment DB, as the ſegment AF is to the ſegment FC. Suppoſe, for example, the ſegment AD to be triple of the ſegment DB; the ſegment AF will be triple of the ſegment FC. Draw the two diagonals DC, FB.

The triangles AFD, DFB are between the ſame parallels; as will be eaſily conceived by ſuppoſing a line drawn through the point F parallel to the ſide AB. Theſe two triangles are therefore to one another 59 as their baſes; and ſince the baſe AD is triple of the baſe DB, the triangle AFD will be triple of the triangle DFB.

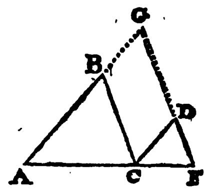
Again,

Again, the triangles BFD, FDC are between the ſame parallels DF, BC, and upon the ſame baſe DF. Theſe two triangles are therefore [31] equal, and ſince the triangle AFD is triple of the triangle DFB, it will alſo be triple of the triangle FDC.

Laſtly, the triangles ADF, FDC are between the ſame parallels; as will be eaſily conceived by ſuppoſing a line drawn through the point D parallel to the ſide AC. Theſe two triangles are therefore to one another
59 as their baſes: and ſince the triangle ADF is triple of the triangle FDC, the baſe AF will be triple of the baſe FC.

PROP. LXI.

Equiangular triangles have their homologous sides proportional.



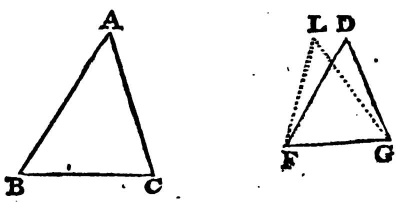
In the two triangles ABC, CDF, if the angle A is equal to the angle C, the angle B equal to the angle D, and the angle C equal to the angle F; the side AC, for example, opposite to the angle B, is to the side CF opposite to the angle D, as the side AB opposite to the angle C, is to the side CD opposite to the angle F. Place the two triangles so that the sides AC, CF shall form one right line; and produce the sides AB, FD, till they meet in G.

The interior and exterior angles GAF, DCF, being equal, the lines GA, DC are [20] parallel. In like manner the alternate angles on the same side GFA, BCA, being equal, the lines GF, BC are [20] parallel. Whence the quadrilateral figure BGDC is a parallelogram, and consequently, its opposite sides are equal. In the triangle GAF, the line BC being parallel to the side GF cuts [60] the other two sides proportionally; that is, AC is to CF, as AB is to BG, or its equal CD.

PROP.

PROP. LXII.

Triangles, the ſides of which are proportional, are equiangular.



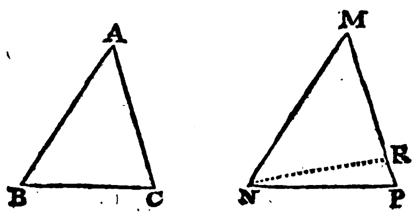
In the two triangles BAC, FDG, if the ſide AB is to the ſide DF, as the ſide BC is to the ſide FG, and as the ſide AC to the ſide DG, theſe two triangles have their angles equal.

Let the ſide AB be ſuppoſed triple of the ſide DF; the ſide AC muſt be triple of the ſide DG, and the ſide BC triple of the ſide FG.

If the triangle FDG be not equiangular with the triangle BAC, another triangle may be formed equiangular with it, for example FLG. But this is impoſſible; for if the two triangles BAC, FLG were equiangular, their ſides would be 61 proportional, and BC being triple of FG, AB would be trip e of LF: but AB is triple of DF; whence LF would be equal to DF. For the ſame reaſon LG would be equal to DG. Thus the two triangles FLG, FDG, having their three ſides equal would be 5 identical; which is abſurd, ſince their angles are unequal.

PROP. LXIII.

Triangles which have an angle in one equal to an angle in the other, and the sides about these angles proportional, are equiangular.



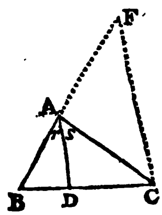
If, in the two triangles BAC, NMP, the angle A be equal to the angle M, and the side AB be to the side MN as the side AC is to the side MP, the two triangles are equiangular.

If AB be triple of MN, AC must be triple of MP. Now, if the angle MNP, for example, is not equal to the angle ABC, another angle may be made, as MNR, which shall be equal to it. But this is impossible: for, the two triangles BAC, NMR having two angles equal, would be equiangular, and consequently 61. would have their sides proportional: whence AB being triple of MN, AC would be triple of MR; which cannot be, since AC is triple of MP.

PROP.

PROP. LXIV.

A right line which biſects any angle of a triangle, divides the ſide oppoſite to the biſected angle into two ſegments, which are proportional to the two other ſides.



In the triangle BAC, let the angle BAC be biſected by the right line AD, making the angle *r* equal to the angle *s*. The ſegment BD is to the ſegment DC, as the ſide BA to the ſide AC.

Produce the ſide BA, and draw CF parallel to DA.

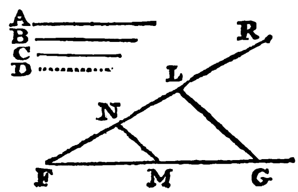
The lines DA, CF being parallel, the interior and exterior angles, *r*, F, are [19] equal, and the alternate angles, *s*, C, are [17] alſo equal. And ſince the angle *r* is equal to the angle *s*, the angle F will alſo be equal to the angle C; and conſequently the ſide AF is equal to the ſide AC.

In the triangle BFC, the line AD being parallel to the ſide FC; BD [60] will be to DC, as BA is to AF or its equal AC.

PROP.

PROP. LXV.

To find a fourth proportional to three given lines.



Let the three lines be A, B, C; it is required to find a fourth line D, ſuch that the line A ſhall be to the line B, as the line C is to the line D.

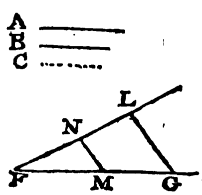
Form any angle RFG; make FM equal to the line A, MG equal to the line B, and FN equal to the line C; draw the right line MN, and through the point G, draw GL parallel to MN; NL will be the fourth proportional required.

In the triangle FLG, the line NM, being parallel to the ſide LG, cuts the other two ſides [60] proportionally. Whence FM is to MG, as FN is to NL; that is A is to B, as C is to D.

PROP.

PROP. LXVI.

To find a third proportional to two given lines.



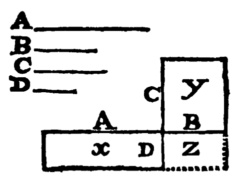
Let the two lines be A, B; it is required to find a third line C, ſuch that the line A ſhall be to the line B, as the line B is to the line C.

Form any angle LFG; make FM equal to the line A, MG equal to the line B, and FN equal to the line B: draw the right line MN, and through the point G draw GL parallel to MN; NL will be the third proportional required.

In the triangle FLG, the line NM being parallel to the ſide LG, cuts the other two ſides [60] proportionally. Whence FM is to MG as FN is to NL; that is, A is to B, as B is to C.

PROP. LXVII.

If four lines be proportional, the rectangle or product of the extremes, is equal to the rectangle or product of the means.



Let the line A be to the line B, as the line C is to the line D; the rectangle formed by the lines A and D, is equal to the rectangle formed by the lines B and C.

Let the four lines meet in a common point, forming at that point four right angles; and draw lines parallel to them to complete the rectangles *x*, *y*, *z*.

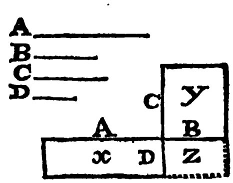
If the line A be triple of the line B, the line C will be triple of the line D.

The rectangles or parallelograms *x*, *z*, being between the same parallels, are to one another as their bases.

bases. Since the base A is triple of the base B, the rectangle x is therefore triple of the rectangle z. In like manner, the rectangles or parallelograms y, z, being between the same parallels, are to one another as their bases: since the base C is triple of the base D, the rectangle y is therefore triple of the rectangle z. Wherefore, the rectangle x being triple of the rectangle z, and the rectangle y being triple of the same rectangle z, these two rectangles x, and y, are equal to one another.

PROP. LXVIII.

Four lines, which have the rectangle or product of the extremes equal to the rectangle or product of the means, are proportional.



Let the four lines A, B, C, D, be ſuch that the rectangle of A and D is equal to the rectangle of B and C; the line A will be to the line B as the line C to the line D.

Let the four lines meet in a common point, forming at that point four right angles; and complete the rectangles x, y, z.

If the line A be triple of the line B, the line C will be triple of the line D.

The rectangles x and z, being between the ſame parallels, are to one another as their baſes. Since the

baſe

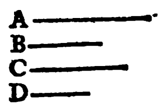
base A is triple of the base B, the rectangle x will therefore be triple of the rectangle z. And the rectangle y is by supposition equal to the rectangle x: the rectangle y is therefore also triple of the rectangle z.

But the rectangles y, z, being between the same parallels, are to one another as their bases: therefore, since the rectangle y is triple of the rectangle z, the base C is also triple of the base D.

PROP.

PROP. LXIX.

If four lines are proportional, they are alſo proportional alternately.



If the line A is to the line B as the line C to the line D; they will be in proportion *alternately*, that is, the line A will be to the line C as the line B to the line D.

Becauſe the line A is to the line B as the line C is to the line D, the rectangle of the extremes, A and D is equal to the rectangle of the means B and C: whence it follows 68 that the line A is to the line C, as the line B to the line D.

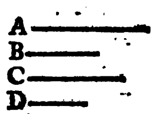
Otherwiſe:

Suppoſing the line A to be triple of the line B, the line C will be triple of the line D. Hence inſtead of ſaying A is to B as C to D, we may ſay, three times B is to B as three times D is to D. Now it is manifeſt that three times B is to three times D, as B is to D. Therefore the line A (which is equal to three times B) is to the line C (which is equal to three times D) as the line B is to the line D.

PROP.

PROP. LXX.

If four lines are proportional, they will be proportional by composition.

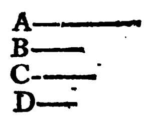


Let the line A be to the line B as the line C is to the line D, they will be proportional by *composition*; that is, the line A joined to the line B, will be to the line B, as the line C joined to the line D, is to the line D.

If the line A contains the line B, for instance, three times, and the line C contains the line D three times; the line A joined to the line B, will contain the line B four times; and the line C joined to the line D, will contain the line D four times. Therefore the line A joined to the line B, is to the line B as the line C joined to the line D, is to the line D.

PROP. LXXI.

If four lines are proportional, they will alſo be proportional by diviſion.

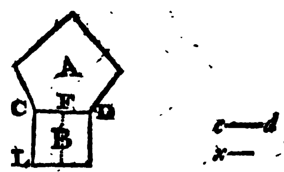


If the line A is to the line B as the line C is to the line D, they will be proportional by *diviſion*, that is, the line A wanting the line B, is to the line B, as the line C wanting the line D, is to the line D.

If the line A contains the line B, for example, three times, and the line C contains the line D three times; the line A wanting the line B, will contain the line B only twice: and the line C wanting the line D, will alſo contain the line D twice. Therefore the line A wanting the line B, is to the line B, as the line C wanting the line D, is to the line D.

PROP. LXXII.

If three lines are proportional, the first is to the third, as the square of the first is to the square of the second.



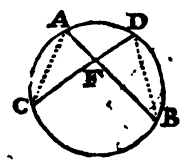
If the line CD is to the line *cd* as the line *cd* is to a third line *x*; the line CD is to the line *x*, as the square of the line CD is to the square of the line *cd*. Take CF equal to the line *x*, and draw the perpendicular FB.

Since the line CD is to the line *cd* as the line *cd* is to the line CF, the rectangle of the extremes CF, CD or CL, is equal 67 to the rectangle of the means, that is to the square of *cd*.

Again, the square of CD, and the rectangle of the lines CF, CL, being between the same parallels, are to one another 58 as their bases. Therefore, CD is to CF, or *x*, as the square of CD is to the rectangle of CF and CL, or to its equal the square of *cd*.

PROP. LXXIII.

If two chords in a circle cut each other, the rectangle of the segments of one is equal to the rectangle of the segments of the other.



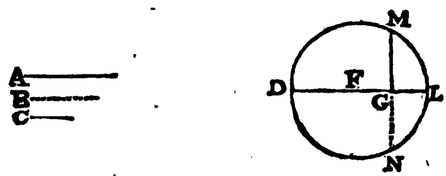
Let the two chords AB, CD in the circle cut each other in the point F, the rectangle of AF, FB is equal to the rectangle of CF, FD. Draw the two right lines AC, DB.

Because in the triangles CAF, BDF the angles at the circumference A and D are both measured 42 by half the arc CB, they are equal. Because the angles C and B are both measured 42 by half the arc AD, these angles are also equal. And the angles at F are equal because they are vertical. These two triangles are therefore equiangular, and consequently, 61 their sides are proportional. Whence the side AF opposite to the angle C, is to the side FD opposite to the angle B, as the side CF opposite to the angle A, is to the side FB opposite to the angle D. Therefore 69 the rectangle of the extremes AF, FB is equal to the rectangle of the means CF, FD.

PROP.

PROP. LXXIV.

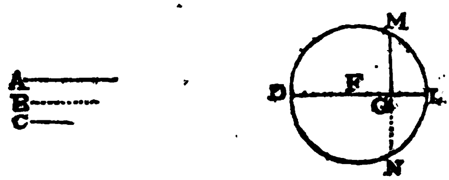
To find a mean proportional between two given lines.



Let there be two lines A, C; it is required to find a third line B, fuch that the line A fhall be to the line B, as the line B is to the line C.

Place the lines A and C in fuch manner that they fhall form one right line DGL; and bifect this right line in the point F. From the point F as a center, defcribe the circumference of a circle DMLN; then, at at the point G, where the two lines are joined, raife the perpendicular GM; GM is the mean proportional fought between the lines A and C. Produce MG to N.

Becaufe the chords DL, MN cut each other at the point G, the rectangle of the fegments DG, GL is 71 equal to the rectangle of the fegments MG, GN.



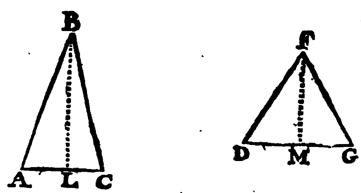
Becaufe the radius FL is perpendicular to the chord MN, FL 38 bifects MN; therefore GN is equal to GM.

Laftly, becaufe the rectangle of the extremes DG, GL is equal to the rectangle of the means GM, GN or its equal GM, DG is to GM, as GM is to GL. Therefore GM is a mean proportional between DG and GL, that is, between the lines A and C.

PROP.

PROP. LXXV.

The bases and altitudes of equal triangles are in reciprocal or inverse ratio.



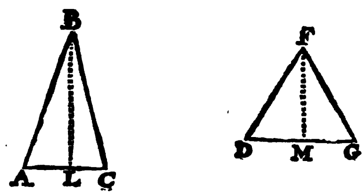
Let the two triangles ABC, DFG be equal: the base AC will be to the base DG, as the perpendicular FM to the perpendicular BL, that is, the bases and altitudes are in *reciprocal* or *inverse ratio*.

The triangle ABC is 54 half the product or rectangle of the base AC and the altitude BL. Again, the triangle DFG is 54 half the product or rectangle of the base DG and the altitude FM. The two triangles being equal, the two rectangles, which are double of the triangles will therefore also be equal.

Again, because the rectangle of the extremes AC, BL, is equal to the rectangle of the means DG, FM, 68 AC is to DG as FM is to BL.

PROP. LXXVI.

Triangles, the bases and altitudes of which are in reciprocal or inverse ratio, are equal.

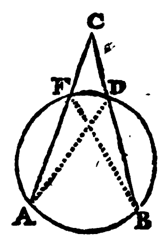


In the two triangles ABC, DFG, if the base AC is to the base DG, as the perpendicular FM to the perpendicular BL, the surfaces of the two triangles are equal.

Because AC is to DG as FM is to BL, the product or rectangle of the extremes AC, BL is [67] equal to the product or rectangle of the means DG, FM. The halves (27. Cor.) of these two rectangles, namely, the triangles ABC, DFG, are therefore equal.

PROP. LXXVII.

Two fecants drawn from the fame point to a circle, are in the inverfe ratio of the parts which lie out of the circle.

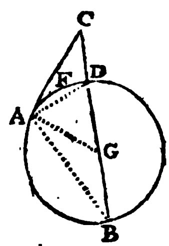


Let the two fecants be CA, CB; CA is to CB, as CD is to CF. Draw the right lines FB, DA.

In the triangles CDA, CFB, the angles at the circumference A and B, being both meafured 42 by half the arc FD, are equal: and the angle C is common to the two triangles. Thefe two triangles are therefore 23 equiangular, and 61 have their fides proportional. Whence the fide CA of the firft triangle, is to the fide CB of the fecond triangle, as the fide CD of the firft triangle is to the fide CF of the fecond triangle.

PROP. LXXVIII.

The tangent to a circle is a mean proportional between the ſecant, and the part of the ſecant which lies out of the circle.



In the circle ABD, CB being ſecant and CA tangent; CB is to CA, as CA is to CD. Draw the right lines AB, AD.

The triangles CAB, CDA have the angle C common to both. Alſo, the angle B is meaſured [42] by half the arc AFD; and the angle CAD, formed by the tangent AC and the chord AD, is meaſured [41] by half the ſame arc AFD. The two triangles CAB, CDA, having then two angles equal, are [23] equiangular, and conſequently [61] have their ſides proportional. Hence the ſide CB of the greater triangle, oppoſite to the angle CAB, is to the ſide CA of the ſmaller triangle oppoſite to the angle D, as

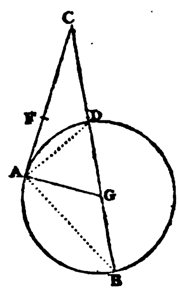
as the side CA of the greater triangle, opposite to the angle B, is to the side CD of the smaller triangle, opposite to the angle A.

COR. This proposition suggests a new method of finding a mean proportional between two given lines. Take CB equal to one of the given lines, and CD equal to the other; bisect DB; from the point of division as a center describe the circumference DAB; and draw the tangent CA: this tangent is a mean proportional between CB and CD, as appears from the proposition.

PROP.

PROP. LXXIX.

To cut a given line in extreme and mean ratio.



Let it be required to divide the line CA in extreme and mean ratio: that is, to divide it in ſuch manner, that the whole line ſhall be to the greater part, as the greater part is to the leſs.

At the extremity A of the line CA, raiſe a perpendicular AG, equal to half the line CA: from the point G as a center, with the radius GA, deſcribe the circumference ADB; draw the line CB through the center; and take CF equal to CD: the line CA will be divided, at the point F in extreme and mean ratio.

Becauſe [78] CB is to CA, as CA is to CD, by diviſion [71] CB wanting CA, or its equal DB, is to CA, as CA wanting CD, or its equal CF, is to CD; that is, CD or CF is to CA as FA is to CD or CF: that is inverſely, CA is to CF, as CF is to FA, or the line AC is cut in extreme and mean ratio.

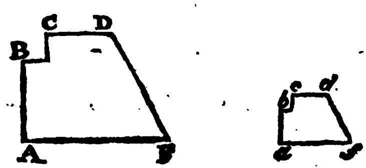
OF

OF

SIMILAR FIGURES.

DEFINITIONS.

1. FIGURES are *Similar*, which are composed of an equal number of physical points disposed in the same



manner. Thus the figures ABCDF, *abcdf* are similar, if every point in the first figure has its corresponding point, placed in the same manner, in the second.

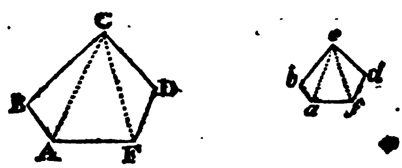
Hence it follows, that if the first figure is, for example, three times greater than the second, the points of which it is composed are three times greater than those of the second figure.

2. In similar figures, those lines are said to be *homologous* which are composed of an equal number of corresponding points.

PROP.

PROP. LXXX.

In ſimilar figures, the homologous lines are proportional.



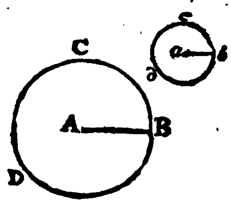
Let the ſimilar figures be ABCDF, *abcdf*, and the homologous lines CA, *ca*; CF, *cf*; CA is to CF, as *ca* is to *cf*.

Since the lines CA, *ca* are homologous, they are compoſed of an equal number of correſponding points; as are alſo the homologous lines CF, *cf*. If, for example, the line CA is compoſed of 40 equal points, and the line CF of 30; the line *ca* will neceſſarily be compoſed of 40 points, and the line *cf* of 30; and it is manifeſt that 40 is to 30, as 40 to 30. Therefore CA is to CF as *ca* to *cf*.

PROP.

PROP. LXXXI.

The circumferences of circles are as their radii.



The circumference DCB is to the radius AB, as the circumference *dcb* is to the radius *ab*.

All circles are similar figures, that is, are composed of an equal number of points, disposed in the same manner: they have therefore so their homologous lines proportional. Therefore the circumference DCB is to the radius AB, as the circumference *dcb* is to the radius *ab*.

PROP. LXXXII.

Similar figures are to each other as the ſquares of their homologous ſides.

Let the two ſimilar figures be A, *a*. Upon the homologous ſides CD, *cd* form the ſquares B, *b*. The ſurface A is to the ſurface *a*, as the ſquare B is to the ſquare *b*.

Since the figures A, *a* are ſimilar, they are compoſed of an equal number of correſponding points; and ſince the homologous ſides CD, *cd*, are compoſed of an equal number of points, the ſquares drawn upon theſe lines, B, *b*, are alſo compoſed of an equal number of points.

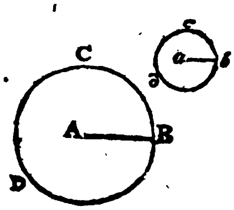
If it be ſuppoſed that the ſurface A is compoſed of 1000 points, and the ſquare B of 400 points; the ſurface *a* will alſo be compoſed of 1000 points and the ſquare *b* of 400. Now it is manifeſt that 1000 is

is to 400, as 1000 to 400. Therefore the ſurface A is to the ſquare B as the ſurface *a* is to the ſquare *b*; and alternately 69 the ſurface A is to the ſurface *a*, as the ſquare B to the ſquare *b*.

COR. Hence it follows, that if any three ſimilar figures be formed upon the three ſides of a right-angled triangle, the figure upon the hypothenuſe will be equal to the other two taken together: for theſe three figures will be as the ſquares of their ſides; therefore, ſince the ſquare of the hypothenuſe is equal to the two ſquares of the other ſides, the figure formed upon the hypothenuſe will alſo be equal to the two other ſimilar figures formed upon the other ſides.

PROP. LXXXIII.

Circles are to each other as the ſquares of their radii.



Let two circles, DCB, *dcb*, be drawn.

The ſurface contained within the circumference DCB, is to the ſurface contained within the circumference *dcb*, as the ſquare formed upon the radius AB to the ſquare formed upon the radius *ab*.

The two circles, being ſimilar figures, are compoſed of an equal number of correſponding points. And the radii AB, *ab*, being compoſed of an equal number of points, the ſquares of theſe radii will alſo be compoſed of an equal number of points.

Suppoſe, for example, that the greater circle DCB is compoſed of 800 points, and the ſquare of the greater

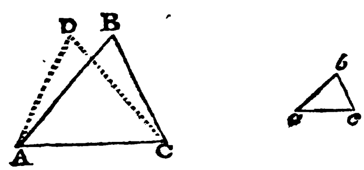
greater radius AB of 300 points, the ſmaller circle *dcb* will alſo be compoſed of 800 points, and the ſquare of the ſmaller radius of 300. Now it is manifeſt that 800 is to 300 as 800 to 300. Therefore the greater circle DCB is to the ſquare of its radius AB, as the ſmaller circle *dcb* is to the ſquare of its radius *ab*; and alternately, the greater circle is to the leſſer circle, as the greater ſquare is to the leſſer ſquare.

PROP. LXXXIV.

Similar triangles are equiangular.

If the two triangles ABC, *abc* be compoſed of an equal number of points diſpoſed in the ſame manner, they are equiangular.

For, ſince the triangles ABC, *abc* are ſimilar figures, they have their ſides [80] proportional: they are therefore [62] equiangular.



PROP. LXXXV.

Equiangular triangles are ſimilar.

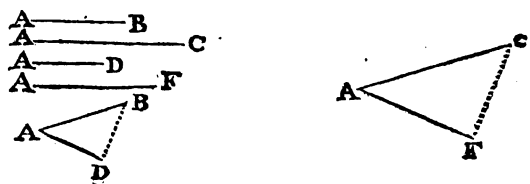
If the triangles ABC, *abc*, are equiangular, they are alſo ſimilar.

If the triangle ABC were not ſimilar to the triangle *abc*, another triangle might be formed upon the line AC, for example ADC, which ſhould be ſimilar to the triangle *abc*. Now the triangle ADC being ſimilar to the triangle *abc*, will alſo [84] be equiangular to *abc*, which is impoſſible, ſince the triangle ABC is ſuppoſed equiangular to *abc*.

PROP.

PROP. LXXXVI.

If four lines are proportional, their squares are also proportional.



If the line AB is to the line AC, as the line AD is to the line AF; the square of the line AB will be to the square of the line AC, as the square of the line AD is to the square of the line AF.

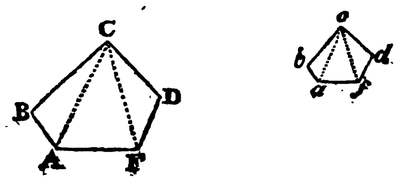
With the lines AB and AD form an angle BAD; with the lines AC and AF form another angle CAF equal to the angle BAD; and draw the right lines BD, CF.

Because AB is to AC as AD to AF, and the contained angles are equal, the two triangles BAD, CAF have their sides about equal angles proportional: they are therefore [63] equiangular, and consequently [85] similar. Whence they are to one another [82] as the squares of their homologous sides. If then the triangle BAD be a third part of the triangle CAF, the square of the side AB will be a third part of the square of the side AC, and the square of the side AD will be a third part of the square of the side AF. Therefore these four squares will be proportional.

PROP.

PROP. LXXXVII.

Similar rectilineal figures may be divided into an equal number of similar triangles,



Let the similar figures be ABCDF, *abcdf*; and draw the homologous lines CA, *ca*; CF, *cf*; these two figures will be divided into an equal number of similar triangles.

The triangles BCA, *bca*, being composed of an equal number of corresponding points, are similar. The triangles ACF, *acf*; and the triangles FCD, *fcd* are also similar for the same reason. Therefore the similar figures ABCDF, *abcdf*, are divided into an equal number of similar triangles.

PROP.

PROP. LXXXVIII.

Similar figures are equiangular.

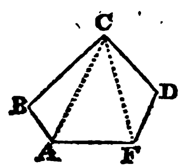
The similar figures ABCDF, *abcdf*, [see fig. on the opposite side] have their angles equal. Draw the homologous lines CA, *ca*; CF, *cf*.

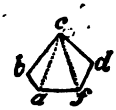
The triangles BCA, *bca* are similar, and consequently equiangular. Therefore the angle B is equal to the angle *b*, the angle BAC to the angle *bac*, and the angle BCA to the angle *bca*. The triangles ACF, *acf*; FCD, *fcd*, are also equiangular because they are similar. Therefore all the angles of the similar figures ABCDF, *abcdf* are equal.

PROP.

PROP. LXXXIX.

Equiangular figures, the sides of which are proportional, are similar.





If the figures ABCDF, *abcdf* have their angles equal, and their sides proportional, they are similar. Draw the right lines CA, *ca*; CF, *cf*.

The triangles CBA, *cba* have two sides proportional and the contained angle equal; they are therefore 63 equiangular, and consequently 85 similar. The lines CA, *ca* are therefore 80 proportional.

The triangles CAF, *caf* have two sides proportional, and the contained angle equal; for, if from the equal angles BAF, *baf* be taken the equal angles BAC, *bac*, there will remain the equal angles CAF, *caf*. These two triangles are therefore equiangular, and consequently similar. In the same manner it may be proved, that the triangles CFD, *cfd* are similar.

The two figures ABCDF, *abcdf* are then composed of an equal number of similar triangles; that is, they are composed of an equal number of points disposed in the same manner, or are similar.

PROP.

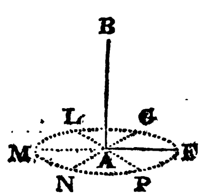
OF

PLANES.

DEFINITIONS.

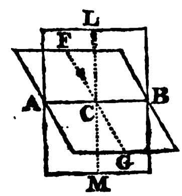
1. A PLANE is a surface such that if a right line, applied to it, touches it in two points, it will touch it in every other point. The surface of a fluid at rest, or of a well-polished table, may be considered as a Plane.

2. A right line is *perpendicular* to a plane, if it makes right angles with all lines which can be drawn from any point in that plane. Thus

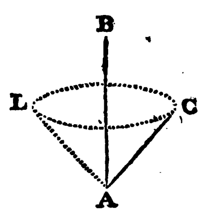


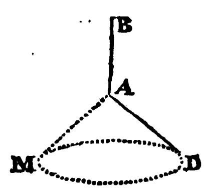
BA is perpendicular to the plane MLGFPN, because it makes right angles with the lines AM, AL, AG, &c. drawn from the point A.

 3. Let



3. Let AB be the common interfection of two planes. If two right lines LM, FG be drawn in thefe two planes perpendicular to the line AB, thefe will form four angles at the point C, which are called the *Inclinations* of the two planes, or the angles formed by the two planes.

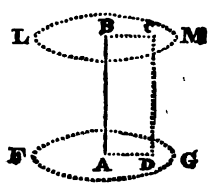




4. If the line AB revolves about itfelf without changing its place, the line AC, which makes an acute angle with AB, will defcribe, in the revolution, a concave furface LAC; and the line AD, which makes an obtufe angle with AB, will defcribe in the revolution a convex furface, MAD.

5. But

5. But the line AF, [see fig. def. 2.] which makes a right angle with AB, will describe in the revolution a surface which will be neither concave nor convex, but plane: and the line AB will be perpendicular to the plane MLGFPN, because it will make right angles with the lines AM, AL, AG, &c. drawn from the point A in that plane.

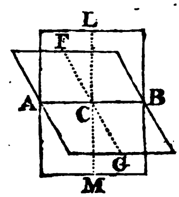


6. Two planes are *Parallel*, when all perpendiculars drawn from one to the other are equal.

PROP.

PROP. XCII.

The common interſection of two planes is a right line.

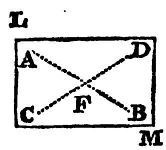


Let the two planes ALBMA, AFBGA interſect each other; the line which is common to both is a right line. Draw a right line from the point A to the point B.

Becauſe the right line AB touches the two planes in the points A and B, it will touch them [def. 1.] in all other points: this line is therefore common to the two planes. Wherefore the common interſection of the two planes is a right line.

PROP. XCIII.

If three points, not in a right line, are common to two planes, these two planes are one and the same plane.



Let two planes be supposed to be placed upon one another, in such manner that the three points, A, B, C shall be common to the two planes; all their other points will also be common, and the two planes will be one and the same plane. The point D, for example, is common to both planes. Draw the right lines AB, CD.

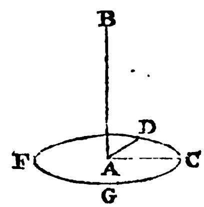
Because the right line AB touches the two planes in the points A and B, it will touch them [def. I.] in every other point; it will therefore touch them in the point F. The point F is therefore common to the two planes.

Again, because the right line CD touches the two planes in the points C and F, it will touch it in the point D: therefore the point D is common to the two planes. The same may be shewn concerning every other point. Wherefore the two planes coincide in all points, or are one and the same plane.

PROP.

PROP. XCIV.

If a right line be perpendicular to two right lines which cut each other, it will be perpendicular to the plane of theſe right lines.



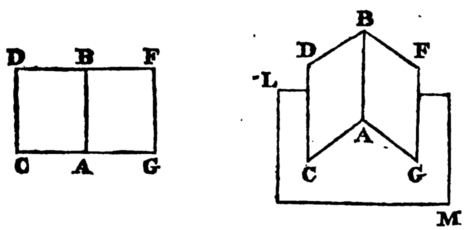
Let the line AB make right angles with the lines AC, AD; it will be perpendicular to the plane which paſſes through theſe lines.

If the line AB were not perpendicular to the plane FDCG, another plane might be made to paſs through the point A, to which the line AB would be perpendicular. But this is impoſſible: for ſince the angles BAC, BAD are right angles, this other plane [def. 2.] muſt paſs through the points C, D; it would therefore 93 be the ſame with the plane FDCG, ſince theſe two planes would have three common points A, C, D.

PROP.

PROP. XCV.

From a given point in a plane, to raiſe a perpendicular to that plane.



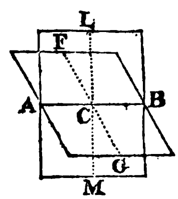
Let it be required to raiſe a perpendicular from the point A in the plane LM.

Form a rectangle CDFG, divide it into two rectangles, having a common ſection AB; and place theſe rectangles upon the plane LM in ſuch manner that the baſes of the two rectangles AC, AG ſhall be in the plane LM, and form any angle with each other; the line AB ſhall be perpendicular to the plane LM.

The line AB makes right angles with the two lines AC, AG, which by ſuppoſition are in the plane LM: it is therefore 94 perpendicular to the plane LM.

PROP. XCVI.

Two planes cutting each other at right angles, if a right line be drawn in one of the planes perpendicular to their common interſection, it will be perpendicular to the other plane.



Let the two planes AFBG, ALBM cut each other at right angles: if the line LC be perpendicular to their common interſection, it is alſo perpendicular to the plane AFBG. Draw CG perpendicular to AB.

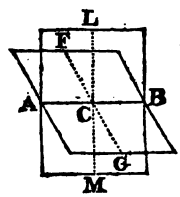
Becauſe the lines CL, CG are perpendicular to the common interſection AB, the angle LCG [def. 3.] is the angle of inclination of the two planes. Since the two planes cut each other perpendicularly, the angle of inclination LCG is therefore a right angle.

And becauſe the line LC is perpendicular to the two lines CA, CG in the plane AFBG, it is 94 perpendicular to the plane AFBG.

PROP.

PROP. XCVII.

If one plane meets another plane, it makes angles with that other plane, which are together equal to two right angles.



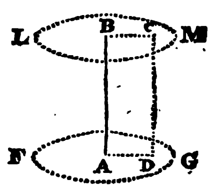
Let the plane ALBM meet the plane AFBG; these planes will make with each other two angles which will together be equal to two right angles. Through any point C, draw the lines FG, LM perpendicular to the line AB.

The line CL makes with the line FG two angles together equal to two right angles. But these two angles are [def. 3.] the angles of inclination of the two planes. Therefore the two planes make angles with each other, which are together equal to two right angles.

COR. It may be demonstrated in the same manner that planes which intersect each other, have their vertical angles equal, that parallel planes have their alternate angles equal, &c.

PROP. XCVIII.

If two planes are parallel to each other, a right line which is perpendicular to one of the planes will alſo be perpendicular to the other.



Let the two planes LM, FG be parallel. If the line BA be perpendicular to the plane FG, it will alſo be perpendicular to the plane LM. From any point C in the plane LM draw CD perpendicular to the plane FG, and draw BC, AD.

Becauſe the lines BA, CD are perpendicular to the plane FG, the angles A, D are right angles.

Becauſe the planes LM, FG are parallel, the perpendiculars AB, DC [def. 6.] are equal; from whence it follows, that the lines BC, AD are parallel.

The

The line BA, being at right angles to the line AD, will alſo [13] be at right angles to the parallel line BC. The line BA is therefore perpendicular to the line BC.

In the ſame manner it may be demonſtrated that the line BA is at right angles to all other lines which can be drawn from the point B in the plane LM. Therefore [def. 2.] the line BA is perpendicular to the plane LM.

OF

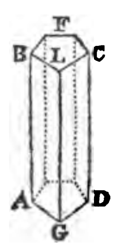
SOLIDS.

DEFINITIONS.

1. A *Solid*, as we have faid, is that which has length, breadth, and thicknefs.

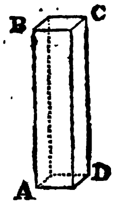
2. A *Polyhedron*, is a folid terminated by plane furfaces.

3. A *Prifm*, is a folid terminated by two identical plane bafes parallel to each other, and by furfaces which are parallelograms.

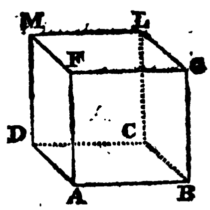


4. A *Paral-*

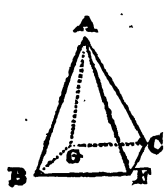
4. A *Parallelopiped*, is a prism, the bases of which are parallelograms.



5. A *Cube*, is a solid terminated by six square surfaces: a die, for example, is a cube.



6. If right lines be raised from every point in the perimeter of any rectilineal figure, and meet in one common point, these lines together with the rectilineal figure, inclose a solid which is called a *Pyramid*.

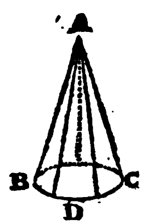


7. A *Cylinder*

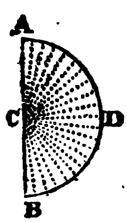
7. A *Cylinder* is a ſolid terminated by two baſes which are equal and parallel circles, and by a convex surface; "or, it is a ſolid formed by the revolution of a parallelogram about one of its ſides."

8. If right lines be raiſed from every point in the circumference of a circle, and meet in one common point, theſe lines together with the circle incloſe a ſolid which is called a *Cone*.

[illegible] a cone

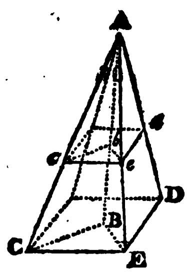


9. A ſemicircle, revolving about its diameter, forms a ſolid which is called a *Sphere*.



10. If

10. If from the vertex of a ſolid, a perpendicular be let fall upon the oppoſite plane, this perpendicular is called the *Altitude* of the ſolid. In the pyramids ACD, A*cd*, AB, A*b* are their reſpective altitudes.



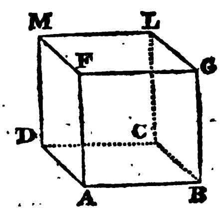
11. Solids are ſaid to be *equal*, if they incloſe an equal ſpace: thus a cone and a pyramid are equal ſolids if the ſpace incloſed within the cone be equal to the ſpace incloſed within the pyramid.

12. *Similar ſolids* are ſuch as conſiſt of an equal number of phyſical points, diſpoſed in the ſame manner.

Thus [in the fig. def. 10.] the larger pyramid ACD, and the ſmaller pyramid A*cd* are ſimilar ſolids, if every point in the larger pyramid has a point correſponding to it in the ſmaller pyramid. A hundred muſket-bullets, and a hundred cannon-balls, diſpoſed in the ſame manner, form two ſimilar ſolids.

PROP. XCIX.

The solid content of a cube is equal to the product of one of its sides twice multiplied by itself.



Let the lines AD, AB be equal. Let the line AD, drawn perpendicular to AB, be supposed to move through the whole length of AB: when it is arrived at BC, and coincides with it, it will have formed the square DABC, and will have been multiplied by the line AB.

Next, let the line AF be drawn equal to AD and perpendicular to the plane DABC; and suppose the plane DABC to move perpendicularly through the whole length of the line AF; when it is arrived at the plane MFGL and coincides with it, it will have formed the

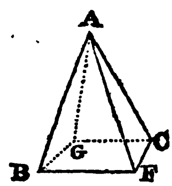
the cube AFLC, and will have been multiplied by the the line AF.

Hence it appears, that to form the cube AFLC, it is neceſſary firſt to multiply the ſide AD by the ſide AB, equal to AD; and then to multiply the product, that is the ſquare of AC, by the ſide AF equal to AD; that is, it is neceſſary to multiply AD by AD, and to multiply the product again by AD.

PROP.

PROP. C.

Similar solids have their homologous lines proportional.



Let the two solids A, *a* be similar; and let their homologous lines be AB, *ab*, BG, *bg*: AB will be to BG, as *ab* to *bg*.

Because the solids A, *a* are similar, every point in the solid A has a point corresponding to it, and disposed in the same manner, in the solid *a*. Thus if the line AB is composed of 20 physical points, and the line BG of 10, the line *ab* will be composed of 20 corresponding points, and the line *bg* of 10. Now it is evident, that 20 is to 10 as 20 to 10: therefore AB is to BG as *ab* to *bg*.

PROP.

PROP. CI.

Similar solids are equiangular.

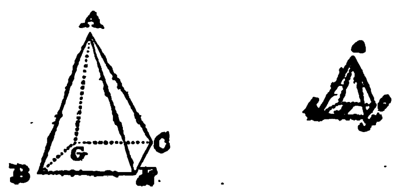
Let the solids [see fig. on the opposite page] A, *a* be similar; their corresponding angles are equal.

Because the solids A, *a*, are similar, the surfaces BAF, *baf*, are composed of an equal number of points disposed in the same manner. These surfaces are therefore similar figures, and consequently 88 equiangular. The angles B, A, F are therefore equal to the angles *b*, *a*, *f*. In the same manner it may be demonstrated that the other correspondent angles are equal.

PROP.

PROP. CII.

Solids which have their angles equal and their ſides proportional, are ſimilar.



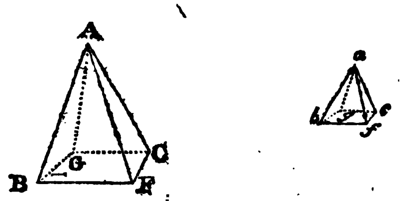
If the ſolids A, *a* have their angles equal and their ſides proportional, they are ſimilar.

For if the ſolids A, *a* were not ſimilar, another ſolid might be formed upon the line BF ſimilar to the ſolid *a*. But this is impoſſible. For, in order to form this other ſolid, ſome angle or ſome ſide of the ſolid A muſt be increaſed or diminiſhed; and then this new ſolid would not have all its angles equal, and all its ſides proportional, to thoſe of the ſolid *a*: that is, 100 101 would not be ſimilar.

PROP.

PROP. CIII.

Similar solids are to one another as the cubes of their homologous sides.

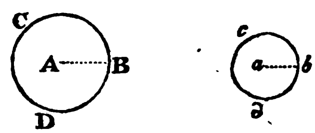


Let A, a be two similar solids: the solid A contains the solid a, as many times as the cube formed upon the side BF contains the cube formed upon the side bf.

Because the solid A is similar to the solid a, every point in the solid A, has its corresponding point in the solid a. From whence it follows that if the side BF is composed, for example, of 50 points, the side bf will also be composed of 50 points: and consequently, the cubes formed upon the sides BF, bf will be composed of an equal number of points.

Let it then be supposed that the solid A is composed of 4000 points, and the cube of the side BF

of 5000 points; the solid *a* must be composed of 4000 points, and the cube of the side *bf* of 5000 points. Now it is evident that 4000 is to 5000 as 4000 to 5000. Therefore the solid A is to the cube of BF, as the solid *a* to the cube of *bf*: and alternately, the solid A is to the solid *a*, as the cube of BF to the cube of *bf*.

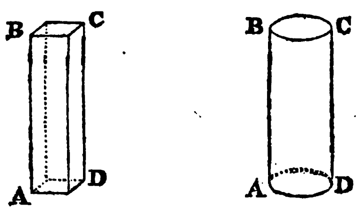


COR. It may be demonstrated in the same manner, that the spheres A, *a*, which are similar solids, are to one another as the cubes of their radii AB, *ab*.

PROP.

PROP. CIV.

The ſolid content of a perpendicular priſm, is equal to the product of its baſe and height.



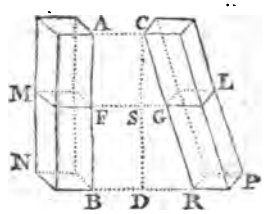
The ſolid content of the perpendicular priſm ABCD, is equal to the product of its baſe AD and height AB.

If the lower baſe AD be ſuppoſed to move perpendicularly along the height AB, till it coincides with the upper baſe BC, it will have formed the priſm ABCD. Now, the baſe AD will have been repeated as many times as there are phyſical points in the height AB. Therefore the ſolid content of the priſm ABCD, is equal to the product of the baſe multiplied by the height.

COR. In the ſame manner it may be demonſtrated, that the ſolid content of the perpendicular cylinder ABCD, is equal to the product of its baſe AD and height AB.

PROP. CV.

The solid content of an inclined prism, is equal to the product of its base and height.



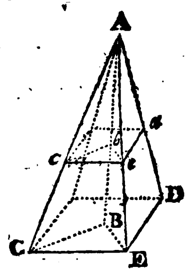
Let the inclined prism be CP; it is equal to the product of its base RP and height CD.

Conceive the base NB of the perpendicular prism NA, and the base RP of the inclined prism PC, to move on, in the same time, parallel to themselves; when they have reached the points A and C, each of them will have been taken over again the same number of times. But the base NB will have been taken over again [104] as many times as there are physical points in the height CD. The base RP will therefore have been taken over again as many times as there are physical points in CD. Consequently the solid content of the inclined prism CP, is equal to the product of its base RP and height CD.

PROP.

PROP. CVI.

In a pyramid, a ſection parallel to the baſe is ſimilar to the baſe.



Let the ſection *cd* be parallel to the baſe CD; this ſection is a figure ſimilar to the baſe. Draw AB perpendicular to the baſe CD; draw alſo BC, *bc*; BE, *be*.

Becauſe the planes *cd*, CD are parallel, AB, being perpendicular to the plane CD, will alſo 98 be perpendicular to the plane *cd*: whence the triangles A*bc*, ABC, having the angles *b*, B right angles, and the angle A common, are equiangular. Therefore 61 A*b* is to AB as *bc* to BC, and as A*c* to AC.

In like manner it may be proved that A*b* is to AB as *be* to BE, and as A*e* to AE. Conſequently, if

A*b*

AB is one third part of AB, bc will be one third part of BC, be the same of BE, Ac of AC, and Ae of AE.

Again, in the two triangles, cAe, CAE there are, about the angle A, common to both, two sides proportional; they are therefore [63] equiangular, and consequently [61] have their other sides proportional. Therefore ce will also be one third part of CE.

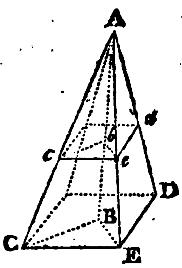
The two triangles cbe, CBE, having their sides proportional, are therefore [89] similar. The same may be demonstrated concerning all the other triangles which form the planes cd, CD. Therefore the section cd is similar to the base CD.

REMARK. If the perpendicular AB falls out of the base, by drawing right lines from the points b, B, it may be demonstrated in the same manner, that the section is similar to the base.

PROP.

PROP. CVII.

In a pyramid, sections parallel to the base are to one another as the squares of their heights.



Let CD, *cd* be parallel sections. From the vertex A draw a perpendicular AB to the plane CD: the plane *cd* is to the plane CD, as the square of the height A*b* is to the square of the height AB. Draw BC, *bc*.

The line AB, being perpendicular to the plane CD, will also 98 be perpendicular to the parallel plane *cd*: whence the angle A*bc* is a right angle, and also the angle ABC: moreover, the angle at A is common to the two triangles A*bc*, ABC: these two triangles are therefore equiangular. Therefore 61 the side *cb* is to the side CB, as the side A*b* to the side AB; and consequently,

ſequently, the ſquare of *cb* is to the ſquare of CB, as the ſquare of A*b* to the ſquare of AB.

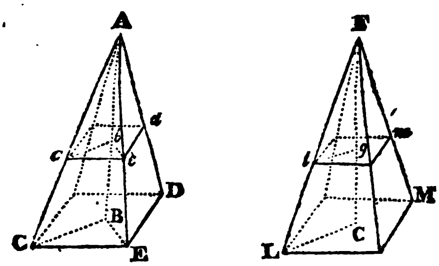
The planes *cd* CD, being [106] ſimilar figures, are to one another [82] as the ſquares of the homologous lines *cb*, CB; they are therefore alſo as the ſquares of the heights A*b*, AB.

COR. In the ſame manner it may be demonſtrated, that in a cone the ſections parallel to the baſe are to one another as the ſquares of the heights or perpendicular diſtances from the vertex.

PROP.

PROP. CVIII.

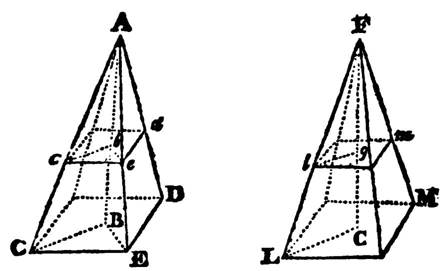
Pyramids of the ſame height are to one another as their baſes.



Let A, F, be two pyramids. If the perpendicular AB is equal to the perpendicular FG, the pyramid A is to the pyramid F, as the baſe CD to the baſe LM. Suppoſing, for example, the baſe CD to be triple of the baſe LM, the pyramid A will be triple of the pyramid F.

Two ſections, *cd*, *lm*, being taken at equal heights A*b*, F*g*; the ſection *cd* is [107] to the baſe CD, as the ſquare of the height A*b* to the ſquare of the height AB: and the ſection *lm* is to the baſe LM, as the ſquare of the height F*g* to the ſquare of the height FG. And becauſe the heights are equal, AB to FG, and A*b* to F*g*, the ſection *cd* is to the baſe CD, as the ſection

lm to the base LM; and alternately, the section *cd* is to the section *lm*, as the base CD is to the base LM. But the base CD is triple of the base LM, therefore the section *cd* is also triple of the section *lm*.



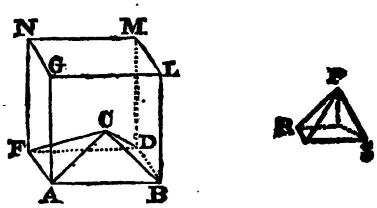
Because the heights AB, FG are equal, it is manifest that the two pyramids are composed of an equal number of physical surfaces placed one upon another. Now it may be demonstrated in the same manner, that every surface or section of the pyramid A is triple of the corresponding surface or section of the pyramid F. Therefore the whole pyramid A is triple of the whole pyramid F.

COR. Hence it follows, that pyramids of the same height and equal bases, are equal; since they are to one another as their bases.

PROP.

PROP. CIX.

A pyramid, whose base is that of a cube, and whose vertex is at the center of the cube, is equal to a third part of the product of its height and base.



Let the cube AM and the pyramid C have the same base AD, and let the vertex of the pyramid be at the center of the cube C; this pyramid is equal to a third part of the product of its height and base.

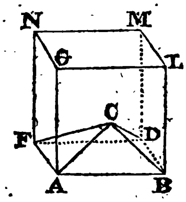
Conceive right lines drawn from the center of the cube to its eight angles A, B, D, F, N, G, L, M; the cube will be divided into six equal pyramids, each of which has one surface of the cube for its base, and half the height of the cube for its height; for example, the pyramid CABDF,

Three of these pyramids will therefore be equal to half the cube. Now the solid content of half the cube is equal to the product of the base and half the height: [compare Prop. 99.] Each pyramid will therefore be equal to one third part of the product of the base and half the height of the cube, that is the whole height of the pyramid.

PROP.

PROP. CX.

The solid content of a pyramid is equal to a third part of the product of its height and base.



Let RPS be a pyramid; its solid content is equal to a third part of the product of its height and its base RS.

Form a cube, the height of which, BL, is double of the height of the pyramid RPS. A pyramid, the base of which is that of this cube, and the vertex of which is C the center of the cube, will be equal to a third part of the product of its base and height.

The pyramids C and P have the same height; they are therefore [Prop. 108. Cor.] to one another as their bases. If the base AFDB is double of the base RS, the pyramid C will therefore be double of the pyramid P.

But the pyramid C is equal to a third part of the product of its height and base. The pyramid P will therefore be equal to a third part of the product of the same height and half the base AFDB, or, which is the same thing, the whole base RS.

PROP.

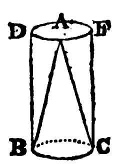
PROP. CXI.

The ſolid content of a cone is equal to a third part of the product of its height and baſe.

For the baſe of a cone may be conſidered as a polygon compoſed of a great number of exceedingly ſmall ſides: and conſequently the cone may be conſidered as a pyramid having a great number of exceedingly ſmall ſurfaces: whence its ſolid content will be equal 110 to one third part of the product of its height and baſe.

PROP. CXII.

The ſolid content of a cone is a third part of the ſolid content of a cylinder deſcribed about it.



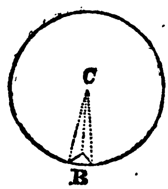
Let the cone BAC, and the cylinder BDFC have the ſame height and baſe, the cone is a third part of the cylinder.

For the cylinder is equal to the product of its height and baſe; and the cone is equal to a third part of this product: therefore the cone is a third part of the cylinder.

PROP.

PROP CXIII.

The ſolid content of a ſphere is equal to a third part of the product of its radius and ſurface.



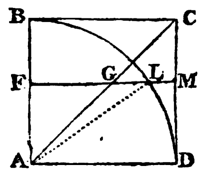
Two points not being ſufficient to make a curve line, three points will not be ſufficient to make a curve ſurface. If therefore all the phyſical points which compoſe the ſurface of the ſphere C be taken, three by three, the whole ſurface will be divided into exceedingly ſmall plane ſurfaces: and radii being drawn to each of theſe points the ſphere will be divided into ſmall pyramids which have their vertex at the center, and have plane baſes.

The ſolid contents of all theſe ſmall pyramids will be equal [110] to a third part of the product of the height and baſes. Therefore the ſolid content of the whole ſphere will be equal to a third part of the product of the height and all the baſes, that is, of its radius and ſurface.

PROP.

PROP. CXIV.

The ſurface of a ſphere is equal to four of its great circles.



If a plane biſect a ſphere, the ſection will paſs through the center, and is called a great circle of the ſphere.

Let ABCD be a ſquare; deſcribe the fourth part of the circumference of a circle BLD; draw the diagonal AC, the right line FM parallel to AD, and the right line AL.

In the triangle ABC, on account of the equal ſides AB, BC, the angles A and C are 4 equal; therefore ſince the angle B is a right angle, the angles A and C are each half a right angle. Again in the triangle AFG, becauſe the angle F is a right angle, and the angle A half a right angle, the angle G is alſo half a right angle: therefore 26 AF is equal to FG.

The

The radius AL is equal to the radius AD; but AD is equal to FM; therefore AL is equal to FM.

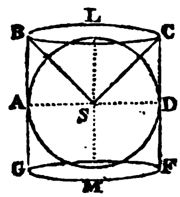
In the rectangular triangle AFL, the square of the hypothenuse AL is equal 32 to the two squares of AF and FL taken together. Instead of AL put its equal FM, and instead of AF put its equal FG; and the square of FM will be equal to the two squares of FG and FL taken together.

Conceive the square ABCD to revolve about the line AB. In the revolution, the square will describe a cylinder, the quadrant a hemisphere, and the triangle ABC an inverted cone, the vertex of which will be in A. Also, the line FM will form a circular section of a cylinder, the line FL will form a circular section of a hemisphere, and the line FG a circular section of a cone.

These circular sections, or circles are to each other 85 as the squares of their radii: therefore, since the square of the radius FM is equal to the squares of the two radii FL and FG, the circular section of the cylinder will be equal to the circular sections of the hemisphere and cone.

In the same manner it may be demonstrated, that all the other sections, or circular surfaces of which the cylinder is composed, are equal to the corresponding sections, or surfaces, of the hemisphere and cone. Therefore the cylinder is equal to the hemisphere and

cone taken together: but the cone [112] is equal to a third part of the cylinder; the hemisphere therefore is equal to the remaining two thirds of the cylinder; and consequently the hemisphere is double of the cone.



The cone BSC is [111] equal to a third part of the product of the radius and the base BC, which is a great circle of the sphere: the hemisphere ALD is therefore equal to a third part of the product of the radius and two of its great circles; and consequently, the whole sphere is equal to a third part of the product of the radius and four of its great circles.

Lastly, since the sphere is equal [113] to a third part of the product of the radius and surface of the sphere, and also to a third part of the product of the radius and four of its great circles, the surface of the sphere is equal to four of its great circles.

THE END.

[illegible]